THE WEIGHTED HERON DUAL MEAN OF SEVERAL POSITIVE NUMBERS

ZHEN-GANG XIAO, V. LOKESHA, AND ZHI-HUA ZHANG

Abstract. In this paper, a definition of the weighted Heron dual mean of several positive numbers is given, its monotonicity is proved, and an identity relating to it is obtained.

1. Introduction

For positive numbers a_0, a_1, let

$$L = L(a_0, a_1) = \begin{cases} \dfrac{a_0 - a_1}{\ln a_0 - \ln a_1}, & a_0 \neq a_1; \\ a_0, & a_0 = a_1; \end{cases} \tag{1.1}$$

$$H = H(a_0, a_1) = \frac{a_0 + \sqrt{a_0 a_1} + a_1}{3}. \tag{1.2}$$

These are respectively called the logarithmic and Heron means (see [1]).

In 2004, Zhang and Wu [2] gave the generalization of Heron mean and its dual form in two variables respectively as follows

$$H(a_0, a_1; k) = \frac{1}{k+1} \sum_{i=0}^{k} a_0^{\frac{k-i}{k}} a_1^{\frac{i}{k}}, \tag{1.3}$$

and

$$h(a_0, a_1; k) = \frac{1}{k} \sum_{i=1}^{k} a_0^{\frac{k+1-i}{k+1}} a_1^{\frac{i}{k+1}}, \tag{1.4}$$

where k is a natural number. Authors proved that $H(a_0, a_1; k)$ is a monotone decreasing function and $h(a_0, a_1; k)$ is a monotone increasing function with k, and

$$\lim_{k \to +\infty} H(a_0, a_1; k) = \lim_{k \to +\infty} h(a_0, a_1; k) = L(a_0, a_1).$$

Let $a = (a_0, a_1, \cdots, a_n)$ and r be a nonnegative integer, where a_i for $0 \le i \le n$ are nonnegative real numbers. Then

$$H_n^{[r]} = H_n^{[r]}(a) = \frac{1}{\binom{n+r}{r}} \sum_{\substack{i_0+i_1+\cdots+i_n=r, \\ i_0, i_1, \cdots, i_n \ge 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k/r} \tag{1.5}$$

and

$$h_n^{[r]} = h_n^{[r]}(a) = \frac{1}{\binom{n+r-1}{r-1}} \sum_{\substack{i_1+i_2+\cdots+i_n=n+r, \\ i_1, i_2, \ldots, i_n \ge 1 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k/(n+r)}, \tag{1.6}$$

are called the generalized Heron mean and its dual form of a, respectively.

In 2003, Zhang and Xiao [3] obtained that for any nonnegative integers r, s with $s > r$, then

$$H_n^{[r]}(a) \geqslant H_n^{[s]}(a), \tag{1.7}$$

and

$$h_n^{[r]}(a) \leqslant h_n^{[s]}(a), \tag{1.8}$$

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with both equalities holding if and only if $a_0 = a_1 = \cdots = a_n$, and

$$\lim_{r\to\infty} H_n^{[r]}(a) = \lim_{r\to\infty} h_n^{[r]}(a) = L(a) = \frac{n!V(\ln a;1,0)}{V(\ln a;0,n)}, \tag{1.9}$$

where $L(a) = \frac{n!V(\ln a;1,0)}{V(\ln a;0,n)}$ is called the logarithmic mean of several positive numbers, and $\ln a = (\ln a_0, \ln a_1, \ldots, \ln a_n)$, $a_i \neq a_j$ for $i \neq j$,

$$V(\ln a;r,k) = \begin{vmatrix} 1 & \ln a_0 & \ln^2 a_0 & \cdots & \ln^{n-1} a_0 & a_0^r \ln^k a_0 \\ 1 & \ln a_1 & \ln^2 a_1 & \cdots & \ln^{n-1} a_1 & a_1^r \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln a_n & \ln^2 a_n & \cdots & \ln^{n-1} a_n & a_n^r \ln^k a_n \end{vmatrix}. \tag{1.10}$$

Let $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and r be a nonnegative integer, where $a_i \geqslant 0$ and $\lambda_i > 0$ for $0 \le i \le n$. Then

$$H_n^{[r]}(a,\lambda) = \frac{1}{\binom{n+r+1}{r}\sum_{i=0}^n \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^n (1+i_k)\lambda_k\right) \prod_{k=0}^n a_k^{i_k/r} \tag{1.11}$$

is called the weighted Heron mean of a for λ.

In [5], authors researched that $H_n^{[r]}(a,\lambda)$ is a monotone decreasing function with k, and

$$H_n^{[r]}(a,\lambda) = \frac{\int_E (\sum_{i=0}^n \lambda_i x_i)(\sum_{i=0}^n a_i^{1/r} x_i)^r dx}{\int_E (\sum_{i=0}^n \lambda_i x_i) dx}, \tag{1.12}$$

and

$$\lim_{r\to\infty} H_n^{[r]}(a,\lambda) = \frac{\int_E (\sum_{i=0}^n \lambda_i x_i)(\prod_{i=0}^n a_i^{x_i}) dx}{\int_E (\sum_{i=0}^n \lambda_i x_i) dx}, \tag{1.13}$$

where $dx = dx_1 dx_2 \cdots dx_n$ denotes the differential of the volume in E :

$$E = \left\{ (x_1, x_2, \cdots, x_n) : \sum_{i=1}^n x_i \leqslant 1, x_i \geqslant 0, i = 1, 2, \cdots, n \right\}, \tag{1.14}$$

and $x_0 = 1 - \sum_{i=1}^n x_i$.

In this paper, a definition of the weighted Heron dual mean of several positive numbers is given, its monotonicity is proved, and an identity relating to it is obtained.

2. Main Results

Definition 2.1. *Let $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and r be a nonnegative integer, where $a_i \geqslant 0$ and $\lambda_i > 0$ for $0 \le i \le n$. Then*

$$h_n^{[r]}(a,\lambda) = \frac{1}{\binom{n+r-1}{r-2}\sum_{i=0}^n \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}} \left(\sum_{k=0}^n (i_k-1)\lambda_k\right) \prod_{k=0}^n a_k^{i_k/(n+r)} \tag{2.1}$$

is called the weighted Heron dual mean of a for λ.

Now, we give some theorems relating to the weighted Heron dual mean $h_n^{[r]}(a)$, the proof of Theorem 2.1 is left to next section.

Theorem 2.1. *If $r \in \mathbb{N}$, then*

$$h_n^{[r]}(a,\lambda) = \frac{\int_E (\sum_{i=0}^n \lambda_i x_i) \left(\sum_{i=0}^n a_i^{1/(-n-r)} x_i\right)^{-n-r} dx}{\int_E (\sum_{i=0}^n \lambda_i x_i) dx}, \tag{2.2}$$

where $dx = dx_1 dx_2 \cdots dx_n$ denotes the differential of the volume in E as (1.14), and $x_0 = 1 - \sum_{i=1}^n x_i$.

Theorem 2.2. *If $r \in \mathbb{N}$, then $h_n^{[r]}(a,\lambda)$ is a monotone increasing function with r, that is*

$$h_n^{[r]}(a,\lambda) \leqslant h_n^{[r+1]}(a,\lambda), \tag{2.3}$$

with equality holding if and only if $a_0 = a_1 = \cdots = a_n$.

Proof. From well-known power mean inequality, we have that

$$M_r(a,x)=\begin{cases}\left(\dfrac{\sum_{k=0}^n a_k^r x_k}{\sum_{k=0}^n x_k}\right)^{\frac{1}{r}}, & r\neq 0;\\ \prod_{k=0}^{n} a_k^{x_k/\sum_{k=0}^n x_k}, & r=0;\end{cases} \tag{2.4}$$

is a monotone increasing function with r, or $M_{1/(-n-r)}(a,x)$ is a monotone increasing function with r.

Combining expression (2.2), we immediately obtain that $h_n^{[r]}(a,\lambda)$ is a monotone increasing function with $r\in\mathbb{N}$. The proof of Theorem 2.2 is completed. □

Theorem 2.3. *If* $r\in\mathbb{N}$, *then*

$$\lim_{r\to\infty} h_n^{[r]}(a,\lambda)=\lim_{r\to\infty} H_n^{[r]}(a,\lambda)=\frac{\int_E(\sum_{i=0}^n\lambda_i x_i)(\prod_{i=0}^n a_i^{x_i})dx}{\int_E(\sum_{i=0}^n\lambda_i x_i)dx}, \tag{2.5}$$

where x_0, dx *and* E *denote as Theorem 2.1.*

Proof. This follows from (2.4), Theorem 2.1 and standard arguments. □

3. The Proof of Theorem 2.1

Throughout this section we assume R is a set of real numbers, and $\mathbb{R}_+$ is a set of strictly positive real numbers. Let $a=(a_0,a_1,\cdots,a_n)\in\mathbb{R}_+^{n+1}$, $a_i\neq a_j$ for $i\neq j$ and φ is a function in $\mathbb{R}$, taking

$$V(a;\varphi)=\begin{vmatrix}1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0)\\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n)\end{vmatrix}. \tag{3.1}$$

Let $\varphi(x)=x^{n+r}\ln^k x$, then we have

$$V(a;r,k)=\begin{vmatrix}1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n+r}\ln^k a_0\\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n+r}\ln^k a_1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^{n+r}\ln^k a_n\end{vmatrix}. \tag{3.2}$$

Note the case $r=0$ and $k=0$ is just the determinant of Van der Monde's matrix of the n-th order:

$$V(a;0{,}0)=\sum_{i=0}^n(-1)^{n+i}a_i^n V_i(a)=\prod_{0\le i<j\le n}(a_j-a_i); \tag{3.3}$$

where

$$V_i(a)=\begin{vmatrix}1 & a_0 & a_0^2 & \cdots & a_0^{n-1}\\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_{i-1} & a_{i-1}^2 & \cdots & a_{i-1}^{n-1}\\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1}\end{vmatrix},\quad (0\le i\le n). \tag{3.4}$$

Also let $0\leqslant i\leqslant n$, we set

$$V(a,i;\varphi)=\begin{vmatrix}1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0)\\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_i & a_i^2 & \cdots & a_i^{n-1} & \varphi(a_i)\\ 0 & 1 & 2a_i & \cdots & (n-1)a_i^{n-2} & \varphi'(a_i)\\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} & \varphi(a_{i+1})\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n)\end{vmatrix}, \tag{3.5}$$

and for $\varphi(x) = x^{n+r+1}$ in (3.5), we have

$$V(a,i;r) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n & a_0^{n+r+1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^n & a_1^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_i & a_i^2 & \cdots & a_i^n & a_i^{n+r+1} \\ 0 & 1 & 2a_i & \cdots & na_i^{n-1} & (n+r+1)a_i^{n+r} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^n & a_{i+1}^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n & a_n^{n+r+1} \end{vmatrix}, \qquad (i \leqslant i \leqslant n). \tag{3.6}$$

and

$$V(a,i;0) = (-1)^{i+1} V(a;0,0) \prod_{j=0, j\neq i}^{n} (a_j - a_i) = (-1)^{i+1} V^2(a;0,0)/V_i(a). \tag{3.7}$$

Lemma 3.1. ([4],[5]) *If $n \in \mathbb{N}$, φ be a $(n+1)$-order differentiable function on interval $\mathbb{I} \subset \mathbb{R}_+$, then we have*

$$V(a;\varphi) = V(a;0,0) \int_E \varphi^{(n)}(A(a,x))dx \tag{3.8}$$

$$\sum_{i=0}^{n} (-1)^{i+1} \lambda_i V(a,i;\varphi) V_i(a) = V^2(a;0,0) \int_E A(\lambda,x) \varphi^{(n+1)}(A(a,x))dx \tag{3.9}$$

where $dx = dx_1 dx_2 \cdots dx_n$ denotes the differential of the volume in E as (1.14), and $a_i, \in \mathbb{I}$, $A(a,x) = a_0 + \sum_{i=1}^{n}(a_i - a_0)x_i = \sum_{i=0}^{n} a_i x_i$, $x_0 = 1 - \sum_{i=1}^{n} x_i$.

Lemma 3.2. *Let r be an integer. Then* (see [4])

$$V(a;r,0) = \begin{cases} V(a;0,0) \cdot \displaystyle\sum_{\substack{i_0+i_1+\cdots+i_n = r, \\ i_0,i_1,\cdots,i_n \geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k}, & r \geqslant 0; \\ 0, & r = -1,-2,\cdots,-n; \\ (-1)^n V(a;0,0) \cdot \displaystyle\sum_{\substack{i_0+i_1+\cdots+i_n = -r, \\ i_0,i_1,\cdots,i_n \geq 1 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k}, & r < -n. \end{cases} \tag{3.10}$$

Lemma 3.3. *Let r be an integer. Then*

$$V^2(a;0,0) E_n^{[r]}(a,\lambda) = \sum_{k=0}^{n} (-1)^{n+k+1} \lambda_k V(a,k;r) V_k(a), \tag{3.11}$$

where

$$E_n^{[r]}(a,\lambda) = \sum_{\substack{i_0+i_1+\cdots+i_n = n+r, \\ i_0,i_1,\cdots,i_n \geq 1 \textit{ are integers}}} \left(\sum_{k=0}^{n} (i_k - 1)\lambda_k \right) \prod_{k=0}^{n} a_k^{-i_k}. \tag{3.12}$$

Proof. Setting a function

$$E_n^{[r]}(a,\lambda) = \sum_{k=0}^{n} \lambda_k B_k(a). \tag{3.13}$$

Let $\lambda_k=1,\lambda_i=0$ $(0\leqslant i\leqslant n, i\neq k)$ in (3.13). Then, from Lemma 3.2 and (3.3), we have

$$\begin{aligned}
B_k &= \sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}} (i_k-1)\prod_{k=0}^{n} a_k^{-i_k}\\
&= \sum_{j=0}^{r}(r-j)a_k^{j-r+1}\sum_{\substack{i_0+i_1+\cdots+i_n=n+j,i_k=1,\\ i_0,i_1,\cdots,i_n\geqslant 1 \text{ are integers}}}\prod_{v=0}^{n} a_v^{-i_v}\\
&= \sum_{j=0}^{r}\left(\sum_{i=0}^{j} a_k^{i-r+1}\right)\sum_{\substack{i_0+i_1+\cdots+i_n=n+j,i_k=1,\\ i_0,i_1,\cdots,i_n\geqslant 1 \text{ are integers}}}\prod_{v=0}^{n} a_v^{-i_v}\\
&= \sum_{j=0}^{r} a_k^{j-r}\left(\sum_{i=0}^{j} a_k^{i-j+1}\sum_{\substack{i_0+i_1+\cdots+i_n=n+j,i_k=1,\\ i_0,i_1,\cdots,i_n\geqslant 1 \text{ are integers}}}\prod_{v=0}^{n} a_v^{-i_v}\right)\\
&= \sum_{j=0}^{r} a_k^{j-r}\sum_{\substack{i_0+i_1+\cdots+i_n=n+j,\\ i_0,i_1,\cdots,i_n\geqslant 1 \text{ are integers}}}\prod_{v=0}^{n} a_v^{-i_v}\\
&= (-1)^n\sum_{j=0}^{r} a_k^{j-r}V(a;-n-j,0)/V(a;0,0)\\
&= \sum_{j=0}^{r} a_k^{j-r}\sum_{i=0}^{n}(-1)^i a_i^{-j}V_i(a)/V(a;0,0)\\
&= \sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{r} a_k^{j-r}a_i^{-j}\right)V_i(a)/V(a;0,0)\\
&= \sum_{i=0,i\neq k}^{n}(-1)^{i+1}\frac{a_k^{-r+1}-a_i^{-r+1}}{a_k-a_i}\frac{V_i(a)}{V(a;0,0)}+(-1)^{k+1}(-r+1)a_k^{-r}\frac{V_k(a)}{V(a;0,0)}.
\end{aligned}$$

That is

$$\begin{aligned}
V^2(a;0,0)B_k &= \sum_{i=0}^{k-1}(-1)^{n+k+i}a_i^{-r+1}\cdot V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\\
&+ a_i^{-r+1}\cdot\left[\sum_{i=0}^{k-1}(-1)^{n+k+i+1}V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+\sum_{i=k+1}^{n}(-1)^{n+k+i}V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\right]\\
&+ \sum_{i=k+1}^{n}(-1)^{n+k+i+1}a_i^{-r+1}\cdot V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+(-1)^{n+k}(-r+1)a_k^{-r}\cdot V(a;0,0)\\
&= (-1)^{n+k}V(a;k,0)V_k(a),
\end{aligned}$$

from (3.13), we know that (3.11) is true. This is proved. □

The Proof of Theorem 2.1. If $r\in\mathbb{N}$ for $r>1$, and taking $\varphi(t)=(-1)^{n+1}\prod_{k=0}^{n}(k+r-1)^{-1}t^{-r+1}$, then $\varphi^{(n+1)}(t)=t^{-(n+r)}$. From Lemma 3.3 and Lemma 3.1, we obtain

$$E_n^{[r]}(a,\lambda)=(-1)^{n+1}\prod_{k=0}^{n}(k+r-1)\int_E A(\lambda,x)A^{-(n+r)}(a,x)dx, \tag{3.14}$$

and

$$\sum_{k=1}^{n}\lambda_k=(n+1)!\int_E A(\lambda,x)dx. \tag{3.15}$$

Let $a^{1/r}=(a_0^{1/r},a_1^{1/r},\cdots,a_n^{1/r})$, $A(a^{1/r},x)=\sum_{k=0}^{n}a_i^{1/r}x_i$, $A(\lambda,x)=\sum_{i=0}^{n}\lambda_i x_i$, we have

$$\begin{aligned}H_n^{[r]}(a) &= \frac{1}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}\sum_{\substack{i_0+i_1+\cdots+i_n=n+r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}}\left(\sum_{k=0}^{n}(i_k-1)\lambda_k\right)\prod_{k=0}^{n}a_k^{i_k/(n+r)}\\ &= \frac{E_n^{[r]}(a^{-1/(n+r)},\lambda)}{\binom{n+r-1}{r-2}\sum_{k=0}^{n}\lambda_k}\\ &= \frac{E_n^{[r]}(a^{-1/(n+r)},\lambda)}{(-1)^{n+1}\prod_{k=0}^{n}(k+r-1)}\cdot\frac{(n+1)!}{\sum_{k=0}^{n}\lambda_k}\\ &= \frac{\int_E\left(\sum_{i=0}^{n}\lambda_i x_i\right)\left(\sum_{i=0}^{n}a_i^{1/(-n-r)}x_i\right)^{-n-r}dx}{\int_E\left(\sum_{i=0}^{n}\lambda_i x_i\right)dx}.\end{aligned}$$

The proof of Theorem 2.1 is completed. □

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(Zh.-G. Xiao) Hunan Institute of Science and Technology, Yueyang, Hunan 414006, China
E-mail address: xiaozg@163.com

(V. Lokesha) Department of Mathematics, Acharya institute of Technology, Soldevahnalli, Hesaragatta Road, Karanataka Bangalore-90 INDIA
E-mail address: lokiv@yahoo.com

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400, China
E-mail address: zxzh1234@163.com