IDENTITIES INVOLVING RATIONAL SUMS

J. L. DÍAZ-BARRERO, J. GIBERGANS-BÁGUENA AND P. G. POPESCU

ABSTRACT. In this note a procedure to obtain identities involving rational sums of real numbers is presented. As an application, some sums involving binomial coefficients and harmonic numbers are obtained.

1. Introduction

Sums involving products of binomial coefficients, rational functions and occasionally harmonic numbers are usually called combinatorial identities though this term ought to include identities involving sums, double sums and other equations too. Many of these sums and techniques to obtain them can be found in the works of Egorychev [1] and Larsen [2]. Recently, in [3] one of these identities was given. Namely,

Theorem 1.1. *Let n be a positive integer. Then, the following identity*

$$\sum_{k=1}^{n} \left\{ (-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{1}{ij} \right\} = \frac{1}{n^2} \tag{1.1}$$

holds.

The goal of this paper is to generalize the preceding result and to present a general procedure to derive identities of this type.

2. Main Results

In what follows we state and prove a general result from which (1.1) immediately follows as a particular case.

Theorem 2.1. *Let n be a positive integer. Then, for all $x \in \mathbb{R}$, holds*

$$\sum_{k=1}^{n} \left\{ (-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{1}{x^2 + (i+j)x + ij} \binom{x+k}{k}^{-1} \right\} = \frac{n}{(x+n)^3}$$

2000 *Mathematics Subject Classification.* 11B15, 26C05.

Key words and phrases. Identities, Combinatorial identities, Finite sums, Binomial coefficients.

Proof. First, we observe that for all $x \in \mathbb{R}$ and $n > 0$, we have

$$\binom{x+n}{n} = \prod_{k=1}^{n} \frac{x+k}{k}.$$

Differentiating both sides of the preceding identity, we get

$$\frac{d}{dx}\binom{x+n}{n} = \binom{x+n}{n} \sum_{j=1}^{n} \frac{1}{x+j} \tag{2.1}$$

Next we claim that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} = \frac{x}{x+n} \tag{2.2}$$

To prove (2.2), we start out with Vandermonde's identity, namely

$$\binom{\alpha}{n}\binom{n}{j} = \binom{\alpha}{j}\binom{\alpha-j}{n-j}$$

and we have

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{\beta}{j}\binom{\alpha}{n}\binom{\alpha}{j}^{-1} = \sum_{j=0}^{n} (-1)^j \binom{\beta}{j}\binom{\alpha-j}{n-j} \tag{2.3}$$

Taking into account Vandermonde's convolution. That is,

$$\sum_{k=0}^{n} \binom{a}{j}\binom{b}{n-j} = \binom{a+b}{n}$$

and the fact that

$$\binom{\alpha}{n} = (-1)^n \binom{n-\alpha-1}{n}$$

then (2.3) becomes

$$\sum_{j=0}^{n} (-1)^j \binom{\beta}{j}\binom{\alpha-j}{n-j} = (-1)^n \sum_{j=0}^{n} \binom{\beta}{j}\binom{n-\alpha-1}{n-j}$$

$$= (-1)^n \binom{\beta-\alpha+n-1}{n} = \binom{\alpha-\beta}{n}.$$

If we set $\alpha = -(x+1)$ into the preceding expression, we get

$$\sum_{j=0}^{n} \binom{n}{j}\binom{\beta}{j}\binom{x+j}{j}^{-1} = \binom{x+\beta+n}{n}\binom{x+n}{n}^{-1}$$

and setting $\beta = -1$ yields

$$\sum_{j=0}^{n} \binom{n}{j} \binom{1}{j} \binom{x+j}{j}^{-1} = \binom{x+n-1}{n} \binom{x+n}{n}^{-1}$$

or equivalently

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} = \frac{x}{x+n}$$

as claimed.

Now differentiating two times (2.2) and taking into account (2.1) yields

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left\{ \left[\left(\sum_{j=1}^{k} \frac{1}{x+j} \right)^2 + \sum_{j=1}^{k} \frac{1}{(x+j)^2} \right] \binom{x+k}{k}^{-1} \right\}$$

$$= \sum_{k=1}^{n} \left\{ (-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{2}{(x+i)(x+j)} \binom{x+k}{k}^{-1} \right\}$$

(2.4)

$$= \sum_{k=1}^{n} \left\{ (-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{2}{x^2 + (i+j)x + ij} \binom{x+k}{k}^{-1} \right\} = \frac{2n}{(x+n)^3}$$

from which the statement immediately follows and the theorem is proven.

□

Setting $x = 0$ into the preceding result, Theorem 1.1 immediately follows. Applying the same procedure, new identities involving finite sums can be obtained. For instance, setting $x = n$ into (2.4), we have

Corollary 2.2. *Let n be a positive integer. Then, holds:*

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \left\{ \left[\left(\sum_{j=1}^{k} \frac{1}{n+j} \right)^2 + \sum_{j=1}^{k} \frac{1}{(n+j)^2} \right] \binom{n+k}{k}^{-1} \right\} = \frac{1}{4n^2}.$$

Now, using Theorem 2.1, we get the following identity.

Theorem 2.3. *Let n be a positive integer. Then, for all $x \in \mathbb{R}$, holds*

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left\{\left[\sum_{j=1}^{k}\frac{1}{(x+j)^3}+\sum_{1\le i<j\le k}\frac{1}{(x+i)(x+j)(2x+i+j)}\right.\right.$$

$$\left.\left.+\sum_{1\le i<j<\ell\le k}\frac{1}{(x+i)(x+j)(x+\ell)}\right]\binom{x+k}{k}^{-1}\right\}=\frac{n}{(n+x)^4}. \qquad (2.5)$$

Proof. Differentiating three times (3) and taking into account (2) we have

$$(-1)\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left\{\left[\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)^3+3\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)\left(\sum_{j=1}^{k}\frac{1}{(x+j)^2}\right)\right.\right.$$

$$\left.\left.+2\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3}\right)\right]\binom{x+k}{k}^{-1}\right\}=\frac{-6n}{(n+x)^4}.$$

Now we observe that

$$\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)^3+3\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)\left(\sum_{j=1}^{k}\frac{1}{(x+j)^2}\right)+2\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3}\right)$$

$$=6\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3}+\sum_{1\le i<j\le k}\frac{1}{(x+i)(x+j)(2x+i+j)}\right.$$

$$\left.+\sum_{1\le i<j<\ell\le k}\frac{1}{(x+i)(x+j)(x+\ell)}\right)$$

and substituting into the preceding expression the proof follows. □

Setting $x = 0$ in identity (2.5), we have

Corollary 2.4. *Let n be a positive integer. Then, holds*

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{j^3}+\sum_{1\le i<j\le k}\frac{1}{ij(i+j)}+\sum_{1\le i<j<\ell\le k}\frac{1}{ij\ell}\right]=\frac{1}{n^3}.$$

References

[1] G. P. Egorychev, *Integral Representation and Computataion of Combinatorial Sums*. Translations of AMS, Vol. 59, American Mathematical Society, 1984.
[2] M. E. Larsen, *Summa Sumarum*. Institut for Matematiske Fag Matematisk Afdeling KØbenhavns Universitet, 2005.
[3] J. L. Díaz-Barrero, “Problem 11164 A Rational Sum”. *Amer. Math. Monthly*, Vol. **112**, No. 6 (2005) 568–568.

Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, 08034 Barcelona. Spain, Automatics and Computer Science Faculty, Politehnica University, Bucureşti, Romania.
E-mail address: jose.luis.diaz@upc.edu
E-mail address: jose.gibergans@upc.edu
E-mail address: pgpopescu@yahoo.com