

# THREE CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING GAMMA AND PSI FUNCTIONS

FENG QI

ABSTRACT. By a simple approach, two classes of functions involving Euler's gamma function and originating from certain problems of traffic flow are proved to be logarithmically completely monotonic and a class of functions involving the psi function is showed to be completely monotonic.

## 1. INTRODUCTION

Recall [12, 22, 23] that a function  $f$  is said to be completely monotonic on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  and  $0 \leq (-1)^n f^{(n)}(x) < \infty$  for  $x \in I$  and  $n \geq 0$ . The set of the completely monotonic functions on  $I$  is denoted by  $\mathcal{C}[I]$ . The well known Bernstein's Theorem [23, p. 161] states that  $f \in \mathcal{C}[(0, \infty)]$  if and only if  $f(x) = \int_0^\infty e^{-xs} d\mu(s)$ , where  $\mu$  is a nonnegative measure on  $[0, \infty)$  such that the integral converges for all  $x > 0$ . This expresses that  $f \in \mathcal{C}[(0, \infty)]$  if and only if  $f$  is a Laplace transform of the measure  $\mu$ .

Recall [1, 6, 12, 15, 17, 19, 20] also that a positive function  $f$  is called logarithmically completely monotonic on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  and its logarithm  $\ln f$  satisfies  $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$  for all  $k \in \mathbb{N}$  on  $I$ . The set of the logarithmically completely monotonic functions on  $I$  is denoted by  $\mathcal{L}[I]$ . In [2, Theorem 1.1] and [6, 20] it is pointed out that the logarithmically completely monotonic functions on  $(0, \infty)$  can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [7, Theorem 4.4].

It was proved in [2, 12, 19, 20, 22] that  $\mathcal{L}[I] \subset \mathcal{C}[I]$ , but not conversely. Stimulated by the papers [17, 19], among other things, it was further revealed in [2] that  $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)]$ , where  $\mathcal{S}$  denotes the set of Stieltjes transforms.

The Kershaw's inequality in [8] states that the double inequality

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \quad (1)$$

holds for  $0 < s < 1$  and  $x \geq 1$ , where  $\Gamma$  denotes the classical Euler's gamma function and  $\psi = \frac{\Gamma'}{\Gamma}$ , the logarithmic derivative of  $\Gamma$ , the psi function. If taking  $s = \frac{1}{2}$  in (1),

---

2000 *Mathematics Subject Classification.* Primary 33B15, 26A48, 26A51; Secondary 26D20.

*Key words and phrases.* logarithmically completely monotonic function, completely monotonic function, gamma function, psi function, inequality.

The author was supported in part by the Science Foundation of the Project for Fostering Innovation Talents at Universities of Henan Province, China.

This paper was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ .

then, for  $x > 1$ ,

$$\sqrt{x + \frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{x + \frac{\sqrt{3}-1}{2}}. \quad (2)$$

Let  $s$  and  $t$  be nonnegative numbers and  $\alpha = \min\{s, t\}$ . For  $x \in (-\alpha, \infty)$ , define

$$z_{s,t}(x) = \begin{cases} \left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} & -x, \quad s \neq t, \\ e^{\psi(x+s)} - x, & s = t. \end{cases} \quad (3)$$

In order to establish the best bounds in Kershaw's inequality (1), among other things, the papers [3, 5, 13, 18] established the following monotonicity and convexity property of  $z_{s,t}(x)$ : The function  $z_{s,t}(x)$  is either convex and decreasing for  $|t-s| < 1$  or concave and increasing for  $|t-s| > 1$ . This result was further generalized in the papers [10, 11].

In [4, p. 123] and [9], while studying certain problems of traffic flow, a double inequality below was obtained for  $n \in \mathbb{N}$ :

$$2\Gamma\left(n + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(n+1) \leq 2^n \Gamma\left(n + \frac{1}{2}\right), \quad (4)$$

In [21], inequality (4) was extended and refined for  $x > 0$  as

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \sqrt{x + \frac{1}{2}}. \quad (5)$$

It is clear that the double inequality (5) is weaker than (2).

Observe that inequality (4) can be rearranged for  $n > 1$  as

$$1 \leq \left[ \frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)} \right]^{1/(n-1)} \leq 2. \quad (6)$$

Hinted by this, the following function  $g(x)$  was defined in [14] for  $x \in (-\frac{1}{2}, \infty)$ :

$$g(x) = \begin{cases} \left[ \frac{\Gamma(1/2)\Gamma(x+1)}{2\Gamma(x+1/2)} \right]^{1/(x-1)}, & x \neq 1, \\ \exp\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right], & x = 1, \end{cases} \quad (7)$$

where  $\gamma = 0.57721566 \dots$  is Euler-Mascheroni's constant, and, among other things, it was proved in [14] that the function  $g(x)$  is logarithmically complete monotonic in  $(-\frac{1}{2}, \infty)$ :  $g(x) \in \mathcal{L}[(-\frac{1}{2}, \infty)]$  with  $\lim_{x \rightarrow -1/2} g(x) = \infty$  and  $\lim_{x \rightarrow \infty} g(x) = 1$ . As consequences of this result, it is deduced that

$$2\Gamma\left(x + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(x+1) \leq 2\Gamma\left(x + \frac{1}{2}\right) \exp\left\{(x-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} \quad (8)$$

for  $x \in [1, \infty)$  and

$$2\Gamma\left(x + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(x+1) \leq 2^x \Gamma\left(x + \frac{1}{2}\right) \quad (9)$$

for  $x \in (0, \infty)$ . It was remarked in [14] that inequalities (8) and (9) extend (4) and (6), the right hand side inequality of (8) refines the right hand side inequality of

(4) and (6), and the right hand side inequalities in (2) and (5) and the following inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \frac{2}{\Gamma(1/2)} \exp\left\{(x-1)\left[1-\gamma-\psi\left(\frac{3}{2}\right)\right]\right\} \quad (10)$$

for  $x \in [1, \infty)$ , which is deduced from the right hand side inequality of (8), are not included with each other respectively.

Now rewrite inequality (4) or (6) for  $n > 1$  as

$$1 \leq \left[ \frac{\Gamma(1+1/2)}{\Gamma(1+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \right]^{1/(n-1)} \leq 2. \quad (11)$$

The definition (7) of  $g(x)$  and inequality (11) motivate us to introduce a new function  $h_\beta(x)$  as follows: Let  $s$  and  $t$  be nonnegative numbers with  $s \neq t$ ,  $\alpha = \min\{s, t\}$  and  $\beta > -\alpha$ . For  $x \in (-\alpha, \infty)$ , define

$$h_\beta(x) = \begin{cases} \left[ \frac{\Gamma(\beta+t)}{\Gamma(\beta+s)} \cdot \frac{\Gamma(x+s)}{\Gamma(x+t)} \right]^{1/(x-\beta)}, & x \neq \beta, \\ \exp[\psi(\beta+s) - \psi(\beta+t)], & x = \beta. \end{cases} \quad (12)$$

The first aim of this paper is to consider the logarithmically completely monotonic property of  $h_\beta(x)$  by a simple approach. Our first main result is the following Theorem 1.

**Theorem 1.** (1) If  $s > t$ , then  $h_\beta(x) \in \mathcal{L}[(-\alpha, \infty)]$  with

$$\lim_{x \rightarrow -\alpha} h_\beta(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} h_\beta(x) = 1. \quad (13)$$

(2) If  $s < t$ , then  $[h_\beta(x)]^{-1} \in \mathcal{L}[(-\alpha, \infty)]$  with

$$\lim_{x \rightarrow -\alpha} h_\beta(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} h_\beta(x) = 1. \quad (14)$$

*Remark 1.* It is noted that taking  $s = 1$ ,  $t = \alpha = \frac{1}{2}$  and  $\beta = 1$  in Theorem 1 can deduce one of the results obtained in [14], the logarithmically complete monotonicity of the function  $g(x)$  defined by (7).

Since  $h_\beta(x)$  is decreasing (or increasing) for  $s > t$  (or  $s < t$ ),  $h_\beta(\beta) = \exp[\psi(\beta+s) - \psi(\beta+t)]$  and  $\lim_{x \rightarrow \infty} h_\beta(x) = 1$ , then the following double inequality (15), as a direct consequence of Theorem 1, is established easily.

**Corollary 1.** Let  $s$  and  $t$  be nonnegative numbers,  $\alpha = \min\{s, t\}$  and  $\beta > -\alpha$ . If  $s > t$ , inequality

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+t)} \leq \frac{\Gamma(x+s)}{\Gamma(x+t)} \leq \frac{\Gamma(\beta+s)}{\Gamma(\beta+t)} \exp\{(x-\beta)[\psi(\beta+s)] - \psi(\beta+t)\} \quad (15)$$

holds for  $x \in [\beta, \infty)$ . If  $s < t$ , inequality (15) reverses.

*Remark 2.* If taking  $\beta = 1$ ,  $s = 1$  and  $t = \frac{1}{2}$  in (15), then inequality (8) is deduced. So, it can be said that inequality (15) is a generalization of (8).

In [16], it was showed that the function  $\ln x - \frac{1}{2x} - \psi(x) \in \mathcal{C}[(0, \infty)]$ . In order to prove Theorem 3 below, this result need to be generalized. Our second main result is the following Theorem 2.

**Theorem 2.** Let  $\alpha \in \mathbb{R}$ . Then  $\psi(x) - \ln x + \frac{\alpha}{x} \in \mathcal{C}[(0, \infty)]$  if and only if  $\alpha \geq 1$  and  $\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}[(0, \infty)]$  if and only if  $\alpha \leq \frac{1}{2}$ .

For  $x \in (0, \infty)$ , define

$$p(x) = \begin{cases} \left[ \frac{x^x}{\Gamma(x+1)} \right]^{1/(1-x)}, & x \neq 1, \\ e^{-\gamma}, & x = 1. \end{cases} \quad (16)$$

In [14], among other things, the logarithmically complete monotonicity of  $p(x)$  was proved:  $p(x) \in \mathcal{L}[(0, \infty)]$  with  $\lim_{x \rightarrow 0+} p(x) = 1$  and  $\lim_{x \rightarrow \infty} p(x) = \frac{1}{e}$ . Motivated by inequality (11) and the definition of  $h_\beta(x)$  in (12), a more general function than  $p(x)$  can be introduced: For  $x \in (0, \infty)$  and  $\alpha > 0$ , let

$$p_\alpha(x) = \begin{cases} \left[ \frac{\Gamma(\alpha+1)}{\alpha^\alpha} \cdot \frac{x^x}{\Gamma(x+1)} \right]^{1/(\alpha-x)}, & x \neq \alpha, \\ \frac{\exp[\psi(\alpha+1) - 1]}{\alpha}, & x = \alpha. \end{cases} \quad (17)$$

It is clear that  $p_1(x) = p(x)$ .

The third aim of this paper is to show the logarithmically completely monotonicity of the function  $p_\alpha(x)$  for any fixed  $\alpha > 0$  by a simple approach. Our third main result is the following Theorem 3.

**Theorem 3.** *For any fixed  $\alpha > 0$ ,  $p_\alpha(x) \in \mathcal{L}[(0, \infty)]$  with*

$$\lim_{x \rightarrow 0+} p_\alpha(x) = \frac{\sqrt[\alpha]{\Gamma(\alpha+1)}}{\alpha} \quad \text{and} \quad \lim_{x \rightarrow \infty} p_\alpha(x) = \frac{1}{e}. \quad (18)$$

## 2. PROOFS OF THEOREMS

It is well-known (see [12, 15, 17, 18, 19, 20]) that, for  $x > 0$  and  $\omega > 0$ ,

$$\frac{1}{x^\omega} = \frac{1}{\Gamma(\omega)} \int_0^\infty t^{\omega-1} e^{-xt} dt, \quad (19)$$

and that, for  $k \in \mathbb{N}$  and  $x > 0$ ,

$$\psi(x) = \ln x + \int_0^\infty \left( \frac{1}{u} - \frac{1}{1-e^{-u}} \right) e^{-xu} du, \quad (20)$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}} dt. \quad (21)$$

*Proof of Theorem 1.* Without loss of generality, assume  $s > t$ . For  $x \neq \beta$ , taking logarithm of the function  $h_\beta(x)$  gives

$$\begin{aligned} \ln h_\beta(x) &= \frac{1}{x-\beta} \left[ \ln \frac{\Gamma(x+s)}{\Gamma(\beta+s)} - \ln \frac{\Gamma(x+t)}{\Gamma(\beta+t)} \right] \\ &= \frac{\ln \Gamma(x+s) - \ln \Gamma(\beta+s)}{x-\beta} - \frac{\ln \Gamma(x+t) - \ln \Gamma(\beta+t)}{x-\beta} \\ &= \frac{1}{x-\beta} \int_\beta^x \psi(u+s) du - \frac{1}{x-\beta} \int_\beta^x \psi(u+t) du \\ &= \frac{1}{x-\beta} \int_\beta^x [\psi(u+s) - \psi(u+t)] du \\ &= \frac{1}{x-\beta} \int_\beta^x \int_t^s \psi'(u+v) dv du \end{aligned}$$

$$\begin{aligned} &\triangleq \frac{1}{x-\beta} \int_{\beta}^x \Phi_{s,t}(u) \, du \\ &= \int_0^1 \Phi_{s,t}((x-\beta)u + \beta) \, du \end{aligned}$$

and, by differentiating  $\ln h_{\beta}(x)$  with respect to  $x$ ,

$$[\ln h_{\beta}(x)]^{(k)} = \int_0^1 u^k \Phi_{s,t}^{(k)}((x-\beta)u + \beta) \, du \quad (22)$$

for  $k \in \mathbb{N}$ .

If  $x = \beta$ , formula (22) is also valid.

Formula (21) implies that  $\psi' \in \mathcal{C}[(0, \infty)]$  and  $\Phi_{s,t}(u) \in \mathcal{C}[(-t, \infty)]$ . This means that  $(-1)^i [\Phi_{s,t}(u)]^{(i)} \geq 0$  holds in  $u \in (-t, \infty)$  for any nonnegative integer  $i$ . Thus,

$$(-1)^k [\ln h_{\beta}(x)]^{(k)} = \int_0^1 u^k \{(-1)^k \Phi_{s,t}^{(k)}((x-\beta)u + \beta)\} \, du \geq 0 \quad (23)$$

in  $(-t, \infty)$  for  $k \in \mathbb{N}$ . The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2.* Formulas (19) and (20) imply that

$$\begin{aligned} \psi(x) - \ln x + \frac{\alpha}{x} &= \int_0^{\infty} \left[ \frac{(1-u)e^u - 1}{u(e^u - 1)} + \alpha \right] e^{-xu} \, du \triangleq \int_0^{\infty} [\theta(u) + \alpha] e^{-xu} \, du, \\ \theta'(u) &= \frac{(u^2 + 2)e^u - 1 - e^{2u}}{u^2(e^u - 1)^2} \triangleq \frac{\theta_1(u)}{u^2(e^u - 1)^2}, \\ \theta_1'(u) &= [u^2 + 2u + 2 - 2e^u]e^u \triangleq e^u \theta_2(u), \\ \theta_2'(u) &= 2(1 + u - e^u) < 0. \end{aligned}$$

It is clear that the function  $\theta_2(u)$  is decreasing in  $(0, \infty)$ . From  $\theta_2(0) = 0$ , it follows that  $\theta_2(u) < 0$  and  $\theta_1'(u) < 0$  in  $(0, \infty)$  and that  $\theta_1'(u)$  decreases in  $(0, \infty)$ . Since  $\theta_1'(0) = 0$ , then  $\theta_1'(u) < 0$  and  $\theta_1(u)$  is decreasing in  $(0, \infty)$ . From  $\theta_1(0) = 0$ , it follows that  $\theta_1(u) < 0$  and  $\theta'(u) < 0$  in  $(0, \infty)$ , and then the function  $\theta(u)$  is decreasing in  $(0, \infty)$ . L'Hôpital's rule yields that  $\lim_{u \rightarrow 0^+} \theta(u) = -\frac{1}{2}$  and  $\lim_{u \rightarrow \infty} \theta(u) = -1$ . Thus,

$$\theta(u) + \alpha \begin{cases} \geq 0, & \alpha \geq 1 \\ \leq 0, & \alpha \leq \frac{1}{2} \end{cases}$$

on  $[0, \infty]$ . Since  $[\psi(x) - \ln x + \frac{\alpha}{x}]^{(k)} = (-1)^k \int_0^{\infty} [\theta(u) + \alpha] u^k e^{-xu} \, du$  for nonnegative integer  $k$ , then  $\psi(x) - \ln x + \frac{\alpha}{x} \in \mathcal{C}[(0, \infty)]$  if  $\alpha \geq 1$  (or  $\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}[(0, \infty)]$  if  $\alpha \leq \frac{1}{2}$ ).

If  $\psi(x) - \ln x + \frac{\alpha}{x} \in \mathcal{C}[(0, \infty)]$  (or  $\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}[(0, \infty)]$ ), then  $\psi(x) - \ln x + \frac{\alpha}{x} > 0$  (or  $< 0$ ), this leads to  $\alpha > x[\ln x - \psi(x)]$  (or  $\alpha < x[\ln x - \psi(x)]$ ) in  $(0, \infty)$ . Since  $\lim_{x \rightarrow 0^+} \{x[\ln x - \psi(x)]\} = 1$  (or  $\lim_{x \rightarrow \infty} \{x[\ln x - \psi(x)]\} = \frac{1}{2}$ ), then  $\alpha \geq 1$  (or  $\alpha \leq \frac{1}{2}$ ). The proof of Theorem 2 is complete.  $\square$

*Proof of Theorem 3.* From the well known differences equation  $\Gamma(x+1) = x\Gamma(x)$ , it follows easily that

$$\psi(x+1) - \psi(x) = \frac{1}{x} \quad (24)$$

for  $x > 0$ . For  $x \neq \alpha$ , taking logarithm of  $p_\alpha(x)$  and using (24) gives

$$\begin{aligned} \ln p_\alpha(x) &= \frac{1}{x-\alpha} \left[ \ln \frac{\Gamma(x+1)}{\Gamma(\alpha+1)} - \frac{x^x}{\alpha^\alpha} \right] \\ &= \frac{\ln \Gamma(x+1) - \ln \Gamma(\alpha+1)}{x-\alpha} - \frac{x \ln x - \alpha \ln \alpha}{x-\alpha} \\ &= \frac{1}{x-\alpha} \int_\alpha^x \psi(u+1) \, du - \frac{1}{x-\alpha} \int_\alpha^x [1 + \ln x] \, du \\ &= \frac{1}{x-\alpha} \int_\alpha^x \left[ \psi(u) - \ln u + \frac{1}{u} - 1 \right] \, du \\ &\triangleq \frac{1}{x-\alpha} \int_\alpha^x [\Psi(u) - 1] \, du \\ &= \int_0^1 \Psi((x-\alpha)u + \alpha) \, du - 1 \end{aligned}$$

and, by differentiating  $\ln p_\alpha(x)$  with respect to  $x$ ,

$$[\ln p_\alpha(x)]^{(k)} = \int_0^1 u^k \Psi^{(k)}((x-\alpha)u + \alpha) \, du \quad (25)$$

for  $k \in \mathbb{N}$ .

If  $x = \alpha$ , formula (25) is also valid.

From Theorem 2, it follows that  $\Psi(u) \in \mathcal{C}[(0, \infty)]$ . This implies  $(-1)^i \Psi^{(i)}(u) \geq 0$  for nonnegative integer  $i$ . As a result, for  $k \in \mathbb{N}$ ,

$$(-1)^k [\ln p_\alpha(x)]^{(k)} = \int_0^1 u^k \{(-1)^k \Psi^{(k)}((x-\alpha)u + \alpha)\} \, du \geq 0 \quad (26)$$

holds in  $x \in (0, \infty)$ . The proof of Theorem 3 is complete.  $\square$

#### REFERENCES

- [1] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.
- [2] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.
- [3] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 100. Available online at <http://jipam.vu.edu.au/article.php?sid=574>.
- [4] M. J. Cloud and B. C. Drachman, *Inequalities with Applications to Engineering*, Springer Verlag, 1998.
- [5] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.
- [6] A. Z. Grinshpan and M. E. H. Ismail, *Completely monotonic functions involving the gamma and  $q$ -gamma functions*, Proc. Amer. Math. Soc. **134** (2006), 1153–1160.
- [7] R. A. Horn, *On infinitely divisible matrices, kernels and functions*, Z. Wahrscheinlichkeitstheorie und Verw. Geb **8** (1967), 219–230.
- [8] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.
- [9] L. Lew, J. Frauenthal and N. Keyfitz, *On the average distances in a circular disc*, in: Mathematical Modeling: Classroom Notes in Applied Mathematics, Philadelphia, SIAM, 1987.
- [10] F. Qi, *A completely monotonic function involving divided differences of psi and polygamma functions and an application*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at <http://rgmia.vu.edu.au/v9n4.html>.

- [11] F. Qi, *A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at <http://rgmia.vu.edu.au/v9n4.html>.
- [12] F. Qi, *Certain logarithmically  $N$ -alternating monotonic functions involving gamma and  $q$ -gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at <http://rgmia.vu.edu.au/v8n3.html>.
- [13] F. Qi, *The best bounds in Kershaw's inequality and two completely monotonic functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at <http://rgmia.vu.edu.au/v9n4.html>.
- [14] F. Qi, J. Cao, and D.-W. Niu, *Four logarithmically completely monotonic functions involving gamma function and originating from problems of traffic flow*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art. 9. Available online at <http://rgmia.vu.edu.au/v9n3.html>.
- [15] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.
- [16] F. Qi, R.-Q. Cui, Ch.-P. Chen, and B.-N. Guo, *Some completely monotonic functions involving polygamma functions and an application*, J. Math. Anal. Appl. **310** (2005), no. 1, 303–308.
- [17] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72. Available online at <http://rgmia.vu.edu.au/v7n1.html>.
- [18] F. Qi, B.-N. Guo, and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436. RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 17. Available online at <http://rgmia.vu.edu.au/v8n2.html>.
- [19] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at <http://rgmia.vu.edu.au/v7n1.html>.
- [20] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **80** (2006), 81–88.
- [21] J. Sándor, *On certain inequalities for the Gamma function*, RGMIA Res. Rep. Coll. **9** (2006), no. 1, Art. 11. Available online at <http://rgmia.vu.edu.au/v9n1.html>.
- [22] H. van Haeringen, *Completely Monotonic and Related Functions*, Report 93-108, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands, 1993.
- [23] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

(F. Qi) COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY, XINXIANG CITY, HENAN PROVINCE, 453007, CHINA; RESEARCH INSTITUTE OF MATHEMATICAL INEQUALITY THEORY, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

*E-mail address:* qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com, 316020821@qq.com

*URL:* <http://rgmia.vu.edu.au/qi.html>