ON AN INEQUALITY OF DIANANDA, III

PENG GAO

Abstract. We extend the results in part I, II on certain inequalities involving the generalized power means.

1. Introduction

Let $M_{n,r}(\mathbf{x})$ be the generalized weighted means: $M_{n,r}(\mathbf{x}) = (\sum_{i=1}^n q_i x_i^r)^{\frac{1}{r}}$, where $M_{n,0}(\mathbf{x})$ denotes the limit of $M_{n,r}(\mathbf{x})$ as $r \to 0^+$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $q_i > 0$ $(1 \leq i \leq n)$ are positive real numbers with $\sum_{i=1}^n q_i = 1$. In this paper, we let $q = \min q_i$ and always assume $n \geq 2, 0 \leq x_1 < x_2 < \cdots < x_n$.

We define $A_n(\mathbf{x}) = M_{n,1}(\mathbf{x}), G_n(\mathbf{x}) = M_{n,0}(\mathbf{x}), H_n(\mathbf{x}) = M_{n,-1}(\mathbf{x})$ and we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x})$, A_n for $A_n(\mathbf{x})$ and similarly for other means when there is no risk of confusion.

For real numbers α, β and mutually distinct numbers r, s, t, we define

$$\Delta_{r,s,t,\alpha,\beta} = \Big| \frac{M_{n,r}^{\alpha} - M_{n,t}^{\alpha}}{M_{n,r}^{\beta} - M_{n,s}^{\beta}} \Big|,$$

where we interpret $M_{n,r}^0 - M_{n,s}^0$ as $\ln M_{n,r} - \ln M_{n,s}$. When $\alpha = \beta$, we define $\Delta_{r,s,t,\alpha}$ to be $\Delta_{r,s,t,\alpha,\alpha}$. We also define $\Delta_{r,s,t}$ to be $\Delta_{r,s,t,1}$.

For $r > s > t \geq 0, \alpha > 0$, we studied in [4] and [5] inequalities of the following two types:

$$C_{r,s,t}\Big((1-q)^{\alpha}\Big) \geq \Delta_{r,s,t,\alpha}, \tag{1.1}$$

and

$$\Delta_{r,s,t,\alpha} \geq C_{r,s,t}(q^{\alpha}), \tag{1.2}$$

where

$$C_{r,s,t}(x) = \frac{1 - x^{1/t-1/r}}{1 - x^{1/s-1/r}}, \quad t > 0; \qquad C_{r,s,0}(x) = \frac{1}{1 - x^{1/s-1/r}}.$$

For any set $\{a, b, c\}$ with a, b, c mutually distinct and non-negative, we let $r = \max\{a, b, c\}, t = \min\{a, b, c\}, s = \{a, b, c\} \backslash \{r, t\}$. By saying (1.1) (resp. (1.2)) holds for the set $\{a, b, c\}, \alpha > 0$ we mean (1.1) (resp. (1.2)) holds for $r > s > t \geq 0, \alpha > 0$. The main result in [5] is the following

Theorem 1.1. *Inequality* (1.1) *holds for the set* $\{r, s, 1\}$ *with* $\alpha = 1$, $r, s, 1$ *mutually distinct and* $r > s \geq 0, r + s \geq 1$. *The equality holds if and only if* $n = 2, x_1 = 0, q_1 = q$.

The consideration of $n = 2, x_2 \to x_1$ shows that inequalities (1.1) and (1.2) can't hold simultaneously in general. By Lemma 2.1 in [5], $C_{r,s,t}(x)$ is an increasing function of x for $0 < x < 1$. Hence in order for (1.1) to hold, it is necessary to have $C_{r,s,t}((\frac{1}{2})^{\alpha}) \geq (r-t)/(r-s)$. In this paper we will complete our discussion on (1.1) for the case $\alpha = 1$ and $1 \in \{r, s, t\}$. We will also show that for $t > 0$, inequality (1.2) doesn't hold.

As an analogue of (1.1), we note the following result of Wu [15]:

Date: January 30, 2006.

1991 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Logarithmic mean, power mean inequality.

Theorem 1.2 ([15, Theorem 2]). *For $x_i > 0, 1 \leq i \leq n$, $\lambda \geq \max\{(n-1)^{p-1}, (p-1)^{p-1}\}, p > 1$,*

$$\Big(\sum_{i=1}^n x_i\Big)^p \leq \lambda \sum_{i=1}^n x_i^p + (n^p - n\lambda)\Big(\prod_{i=1}^n x_i\Big)^{\frac{p}{n}}.$$

The case $1 < p \leq n$ in the above theorem is of particular interest to us and we will give another proof of this case in Section 3. We will also use the idea in Wu's proof of Theorem 1.2 to obtain results of similar kinds.

The one-parameter mean value family

$$L_r = L_r(x,y) = \Big(\frac{x^r - y^r}{r(x-y)}\Big)^{1/(r-1)} \qquad (r \neq 0, 1; x, y > 0, x \neq y)$$

is known as Stolarsky's generalized logarithmic mean [12]. We note here the limit relations

$$\begin{aligned} \lim_{r \to 0} L_r(x,y) &= L(x,y) = \frac{x-y}{\log x - \log y}; \\ \lim_{r \to 1} L_r(x,y) &= I(x,y) = \frac{1}{e}\Big(\frac{x^x}{y^y}\Big)^{1/(x-y)}, \end{aligned}$$

are called the logarithmic and identric means respectively. We refer the reader to the paper [2] and the references therein for many inequalities involving L, I and the generalized power means. Some of these inequalities can be regarded as analogues of inequalities (1.1) and (1.2) considered here. We will derive several inequalities involving the L_r's by applying (1.1) and (1.2) in the last part of this paper.

2. An extension of Theorem 1.1

Theorem 2.1. *Let $1 > r > s \geq 0$. If $C_{1,r,s}(\frac{1}{2}) \geq (1-s)/(1-r)$, then inequality* (1.1) *holds for the set $\{1, r, s\}$ with $\alpha = 1$.*

Proof. The case $s = 0$ was treated in [4] and the case $r + s \geq 1$ was treated in [5], so we may assume $s > 0$ and $r + s < 1$ from now on. Define

$$D_n(\mathbf{x}) = A_n - M_{n,r} - C(1-q)(A_n - M_{n,s}), \ C(x) = \frac{1 - x^{1/r-1}}{1 - x^{1/s-1}}.$$

We need to show $D_n \geq 0$ and we have

$$\frac{1}{q_n}\frac{\partial D_n}{\partial x_n} = 1 - M_{n,r}^{1-r}x_n^{r-1} - C(1-q)(1 - M_{n,s}^{1-s}x_n^{s-1}). \tag{2.1}$$

By a change of variables: $\frac{x_i}{x_n} \to x_i, 1 \leq i \leq n$, we may assume $0 \leq x_1 < x_2 < \cdots < x_n = 1$ in (2.1) and rewrite it as

$$g_n(x_1, \ldots, x_{n-1}) := 1 - M_{n,r}^{1-r} - C(1-q)(1 - M_{n,s}^{1-s}). \tag{2.2}$$

We want to show $g_n \geq 0$. Let $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of g_n is reached. We may assume $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If $a_i = a_{i+1}$ for some $1 \leq i \leq n-2$ or $a_{n-1} = 1$, by combining a_i with a_{i+1} and q_i with q_{i+1} or a_{n-1} with 1 and q_{n-1} with q_n, it follows from Lemma 2.1 of [5] that we can reduce the determination of the absolute minimum of g_n to that of g_{n-1} with different weights. Thus without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$.

If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, and we can regard g_n as a function of $a_2, \ldots, a_{n-1}$, then we obtain

$$\nabla g_n(a_2, \ldots, a_{n-1}) = 0.$$

Otherwise $a_1 > 0$, $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$ and

$$\nabla g_n(a_1, \ldots, a_{n-1}) = 0.$$

In either case $a_2, \ldots, a_{n-1}$ solve the equation

$$(r-1)M_{n,r}^{1-2r}x^{r-1} + C(1-q)(1-s)M_{n,s}^{1-2s}x^{s-1} = 0.$$

The above equation has at most one root (regarding $M_{n,r}, M_{n,s}$ as constants), so we only need to show $g_n \geq 0$ for the case $n = 3$ with $0 = a_1 < a_2 = x < a_3 = 1$ in (2.2). In this case we regard g_3 as a function of x and we get

$$\frac{1}{q_2}g_3'(x) = M_{3,r}^{1-2r}x^{r-1}h(x),$$

where

$$h(x) = r - 1 + (1-s)C(1-q)(q_2x^{s/2} + q_3x^{-s/2})^{(1-2s)/s}(q_2x^{r/2} + q_3x^{-r/2})^{(2r-1)/r}.$$

If $q_2 = 0$ (note $q_3 > 0$), then

$$h(x) = r - 1 + (1-s)C(1-q)q_3^{1/s-1/r}x^{s-r}.$$

One easily checks that in this case $h(x)$ has exactly one root in $(0,1)$. Now assume $q_2 > 0$, then

$$h'(x) = (1-s)C(1-q)M_{3,s}^{1-3s}M_{3,r}^{r-1}x^{-\frac{r+s+2}{2}}p(x),$$

where

$$p(x) = (r-s)(q_2^2x^{r+s} - q_3^2) + (r+s-1)q_2q_3(x^r - x^s).$$

Now

$$p'(x) = x^{r+s-1}\Big((r^2-s^2)q_2^2 + (r+s-1)q_2q_3(rx^{-s} - sx^{-r})\Big) := x^{r+s-1}q(x).$$

It's easy to see that the only positive root for $q'(x) = 0$ is $x = 1$. Thus $p'(x) = 0$ can have at most one root in $(0,1)$. This means that $p(x) = 0$ has at most two roots in $(0,1)$. Now if $q_2 > q_3$, then it follows from this and that $p(0) < 0, p(1) > 0$ that $p(x) = 0$ hence $h'(x) = 0$ has only one root in $(0,1)$. Thus $h(x) = 0$ can have at most two positive roots in $(0,1]$. Note that $\lim_{x\to 0^+} h(x) = +\infty$ and by our assumption that $h(1) \leq 0$. This means that $h(x)$ has at most one root in $(0,1)$ and hence $g_3'(x)$ has at most one root x_0 in $(0,1)$. Since $\lim_{x\to 0^+} g_3'(x) = +\infty$, $g_3(x)$ takes its maximum value at x_0 and we conclude that $g_3(x) \geq \min\{g_3(0), g_3(1)\} = 0$.

Suppose now $q_2 \leq q_3$. By setting $y = q_3/q_2$, we can rewrite $p(x)$ as

$$p(x) = q_2^2\Big((r-s)(x^{r+s} - y^2) + (r+s-1)y(x^r - x^s)\Big) := q_2^2f(y).$$

We now want to show $f(y) \leq 0$ for $y \geq 1, 0 \leq x \leq 1$. We may assume $x > 0$ and note that for fixed x, $f(y)$ is a quadratic function with $f(0) > 0$. Since the coefficient of y^2 is negative, it thus suffices to show that $f(1) \leq 0$. Equivalently, we need to show $p(x) \leq 0$ when $q_2 = q_3$ and by repeating the argument in the preceding paragraph, we see that $p'(x) = 0$ has no root in $(0,1)$, hence $p(x) \leq \max\{p(0), p(1)\} \leq 0$. Thus in this case $h(x)$ is a decreasing function of x for $0 \leq x \leq 1$. It follows from this that $h(x) = 0$ has only one root in $(0,1]$ and similarly to the argument in the preceding paragraph, we have $g_3(x) \geq \min\{g_3(0), g_3(1)\} = 0$.

Thus we have shown $g_n \geq 0$, hence $\frac{\partial D_n}{\partial x_n} \geq 0$ with equality holding if and only if $n = 1$ or $n = 2, x_1 = 0, q_1 = q$. By letting x_n tend to x_{n-1}, we have $D_n \geq D_{n-1}$ (with weights $q_1, \ldots, q_{n-2}, q_{n-1} + q_n$). Since $C(1-q)$ is an increasing function of q by Lemma 2.1 of [5], it follows by induction that $D_n > D_{n-1} > \cdots > D_2 = 0$ when $x_1 = 0, q_1 = q$ in D_2. Else $D_n > D_{n-1} > \cdots > D_1 = 0$ and this completes the proof. □

Now we show that inequality (1.2) doesn't hold for $t > 0$. It suffices to consider the case $n = 2$ with $x_2 = 1, q_2 = q$. We set $x_1 = x$ and define

$$f(x) = M_{2,r}^{\alpha} - M_{2,t}^{\alpha} - C_{r,s,t}(q^{\alpha})(M_{2,r}^{\alpha} - M_{2,s}^{\alpha}).$$

Note $f(0)=0$ and

$$\frac{f'(x)}{\alpha(1-q)x^{t-1}}=M^{\alpha-r}_{2,r}x^{r-t}-M^{\alpha-t}_{2,t}-C_{r,s,t}(q^{\alpha})(M^{\alpha-r}_{2,r}x^{r-t}-M^{\alpha-s}_{2,s}x^{s-t}).$$

If $t>0$, the right-hand side above is negative when $x=0$, which implies $f(x)<0$ for positive x small enough so that (1.2) fails to hold.

In [5], the author asked whether it is true or not that if (1.1) holds for $r>s>t\geq 0, \alpha>0$, then it also holds for $r>s>t\geq 0, k\alpha$ with $0<k<1$ and if (1.2) holds for $r>s>t\geq 0, \alpha>0$, then it also holds for $r>s>t\geq 0, k\alpha$ with $k>1$. Now the second part above is true due to the discussion above since (1.2) only holds when $t=0$ and this case has already been discussed in Theorem 3.2 of [5]. We would also like to take the opportunity to correct some typos in the statement of Theorem 3.2 in [5], namely, one needs to exchange the two conditions $k>1$ and $0<k<1$ there. We now proceed to show that the first part is actually not true.

We write for convenience that

$$C=C_{r,s,t}\Big((1-q)^{\alpha}\Big),\quad C'=C_{r,s,t}\Big((1-q)^{k\alpha}\Big),$$

so that we can rewrite (1.1) as

$$M^{\alpha}_{n,s}\leq(1-\frac{1}{C})M^{\alpha}_{n,r}+\frac{1}{C}M^{\alpha}_{n,t}.$$

Now, what we assert is that it follows from above that for $0<k<1$:

$$M^{k\alpha}_{n,s}\leq(1-\frac{1}{C'})M^{k\alpha}_{n,r}+\frac{1}{C'}M^{k\alpha}_{n,t}.$$

Thus to show the assertion doesn't hold, it suffices to find an example such that

$$\Big((1-\frac{1}{C})M^{\alpha}_{n,r}+\frac{1}{C}M^{\alpha}_{n,t}\Big)^k>(1-\frac{1}{C'})M^{k\alpha}_{n,r}+\frac{1}{C'}M^{k\alpha}_{n,t}.$$

Now we set $y=M^{\alpha}_{n,r}/M^{\alpha}_{n,t}$ and note that in the case $n=2, x_1=0, q_2=q$, $y=q^{\alpha(1/r-1/t)}$. Thus it suffices to show that for $y=q^{\alpha(1/r-1/t)}$,

$$\left(\left(1-\frac{1}{C}\right)y+\frac{1}{C}\right)^k>(1-\frac{1}{C'})y^k+\frac{1}{C'}.$$

Let

$$f(y)=\frac{(1-\frac{1}{C'})y^k+\frac{1}{C'}}{\left(\left(1-\frac{1}{C}\right)y+\frac{1}{C}\right)^k}$$

and we want to show $f(q^{\alpha(1/r-1/t)})<1$. Using the definitions for C, C' and the notation

$$\beta=\alpha(1/t-1/r),\quad \gamma=\frac{1/s-1/r}{1/t-1/r},\quad A=\frac{q^{\beta}-1}{1-(1-q)^{\beta}},\quad A'=\frac{q^{k\beta}-1}{1-(1-q)^{k\beta}},\quad z=(1-q)^{\beta\gamma}.$$

We obtain

$$f\Big(q^{\alpha(1/r-1/t)}\Big)=\frac{1+A'(1-z^k)}{(1+A(1-z))^k}:=g(z).$$

We note first here that $A, A'\leq -1$ and $0\leq\gamma\leq 1$. It follows that $(1-q)^{\beta}\leq z\leq 1$. Now

$$g'(z)=\frac{k\Big(-A'(1+A)z^{k-1}+A(1+A')\Big)}{(1+A(1-z))^{k+1}}.$$

The function

$$z\mapsto -A'(1+A)z^{k-1}+A(1+A')$$

only has one positive root and is < 0 when $z \to 0^+$ as long as $q \neq 1/2$. It follows from this and that $g((1-q)^{\beta}) = g(1) = 1$ that $g(z) < 1$ for $(1-q)^{\beta} < z < 1$ if $q \neq 1/2$, which is just what is desired.

3. Another look at Theorem 1.2

Let $k \in \{0, 1, \dots, n\}$, the k-th symmetric function $E_{n,k}$ of $\mathbf{x}$ and its mean $P_{n,k}$ are defined by

$$E_{n,k}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}, 1 \leq k \leq n; \quad E_{n,0} = 1; \quad P^k_{n,k}(\mathbf{x}) = \frac{E_{n,k}(\mathbf{x})}{\binom{n}{k}}.$$

The following lemma is due to Wu, Wang and Fu [14] (see also [1, p. 317-318]):

Lemma 3.1. *Let $2 \leq k \leq n, \mathbf{x} = (x_1, \dots, x_n), x_1 \leq x_2 \leq \cdots \leq x_n$. There exists $\mathbf{y} = (y_1, \dots, y_k)$ with $x_1 \leq y_1 \leq \cdots \leq y_k \leq x_n$ such that $P_{n,i}(\mathbf{x}) = P_{k,i}(\mathbf{y}), 0 \leq i \leq k$. Moreover, if $x_1, \dots, x_n$ are not all equal, then $y_1, \dots, y_k$ are also not all equal.*

In the section, we would like to first give another proof of Theorem 1.2 for the case $1 < p \leq n$ and we will need a result of Shen [11, Lemma 1], which we shall state as

Lemma 3.2. *Let $0 \leq x_i \leq 1$, then for $1 \leq k \leq n$,*

$$P^{k-1}_{n,k-1}(\mathbf{x}) \leq \frac{1}{k} + \frac{k-1}{k} P^k_{n,k}(\mathbf{x}).$$

Proof. By Lemma 3.1, there exists $\mathbf{y} = (y_1, \dots, y_k)$ with $0 \leq x_1 \leq y_1 \leq \cdots \leq y_k \leq x_n$ such that $P_{n,k-1}(\mathbf{x}) = P_{k,k-1}(\mathbf{y})$ and $P_{n,k}(\mathbf{x}) = P_{k,k}(\mathbf{y})$. It thus suffices to prove the lemma for the case $n = k$ and one checks easily this follows from case (ii) of [11, Lemma 1]. □

One can then easily deduce from above the original result of Shen [11, Lemma 1] which we will need, namely, for $0 \leq x_i \leq 1$,

$$P_{n,1}(\mathbf{x}) \leq \frac{n-1}{n} + \frac{1}{n} P^n_{n,n}(\mathbf{x}). \tag{3.1}$$

We now use Lemma 3.2 to deduce

Corollary 3.1. *For $1 < p \leq n$,*

$$\Big(\sum_{i=1}^{n} x_i\Big)^p \leq (n-1)^{p-1} \sum_{i=1}^{n} x_i^p + n(n^{p-1} - (n-1)^{p-1}) \Big(\prod_{i=1}^{n} x_i\Big)^{\frac{p}{n}}. \tag{3.2}$$

Proof. In this proof we assume that $0 \leq x_1 \leq \cdots \leq x_n$ and we define

$$f(\mathbf{x}) = (n-1)^{p-1} \sum_{i=1}^{n} x_i^p + n(n^{p-1} - (n-1)^{p-1}) \Big(\prod_{i=1}^{n} x_i\Big)^{\frac{p}{n}} - \Big(\sum_{i=1}^{n} x_i\Big)^p.$$

If $x_n = 0$ or $x_1 = \cdots = x_n$ then $f = 0$ otherwise we may assume $n \geq 2$ and $x_{k-1} < x_k = \cdots = x_n = x$ for some $1 \leq k \leq n$ (we set $x_0 = 0$ here), then

$$\frac{1}{(n-k+1)px^{p-1}} \frac{\partial f}{\partial x} = (n-1)^{p-1} + (n^{p-1} - (n-1)^{p-1}) \Big(\prod_{i=1}^{n} \frac{x_i}{x}\Big)^{\frac{p}{n}} - \Big(\sum_{i=1}^{n} \frac{x_i}{x}\Big)^{p-1}.$$

We want to show that the right-hand side above is non-negative. Note that $0 \leq x_i/x \leq 1$ so by a change of variables $x_i/x \to x_i$, it suffices to show that

$$(n-1)^{p-1} + (n^{p-1} - (n-1)^{p-1}) \Big(\prod_{i=1}^{n} x_i\Big)^{\frac{p}{n}} - \Big(\sum_{i=1}^{n} x_i\Big)^{p-1} \geq 0$$

for $0 \leq x_i \leq 1$ and $1 < p \leq n$. By (3.1), we only need to show that

$$\Big(n-1+\prod_{i=1}^n x_i\Big)^{p-1} \leq (n-1)^{p-1}+(n^{p-1}-(n-1)^{p-1})\Big(\prod_{i=1}^n x_i\Big)^{\frac{p}{n}}.$$

By further setting

$$t=\Big(\prod_{i=1}^n x_i\Big)^{\frac{p}{n}},$$

it suffices to show that

$$g(t)=(n-1)^{p-1}+(n^{p-1}-(n-1)^{p-1})t-\Big(n-1+t^{n/p}\Big)^{p-1} \geq 0,$$

for $0 \leq t \leq 1$. One checks easily that

$$g''(t)=-\frac{n(p-1)}{p}\Big(n-1+t^{n/p}\Big)^{p-3}t^{\frac{n}{p}-2}h(t),$$

where

$$h(t)=(n-\frac{n}{p}-1)t^{n/p}+(\frac{n}{p}-1)(n-1) \geq \min\{h(0),h(1)\}=\min\{\frac{n}{p}(n-2),(\frac{n}{p}-1)(n-1)\} \geq 0.$$

We conclude that $g(t)$ is a concave function for $0 \leq t \leq 1$. Hence

$$g(t) \geq \min\{g(0),g(1)\}=0.$$

It follows from this that $\frac{\partial f}{\partial x} \geq 0$ and by letting $x \rightarrow x_{k-1}$ and repeating the above argument, we conclude that $f(\mathbf{x}) \geq f(x_1,x_1,\ldots,x_1)=0$ which completes the proof. □

We note here it is pointed out in [6] that Corollary 3.1 was obtained by Chen and Wang in a paper published in a Chinese journal (see the citation [4] in [6]), however, we have no access to the paper for the method used in their proof. We also note that by a change of variables: $x_i \rightarrow x_i^{1/p}$, (3.2) is equivalent to the case $r=1,s=1/p,t=0,\alpha=1$ of (1.1) and this certainly improves Theorem 3.1 in [4].

Similarly, in another paper of Chen and Wang (see the citation [2] in [6]), they have shown that for $n \geq 2, p \geq n/(n-1)$,

$$\Big(\sum_{i=1}^n x_i\Big)^p \geq \sum_{i=1}^n x_i^p+(n^p-n)\Big(\prod_{i=1}^n x_i\Big)^{p/n}. \tag{3.3}$$

Again, by a change of variables: $x_i \rightarrow x_i^{1/p}$, this is equivalent to the case $r=1,s=1/p,t=0,\alpha=1$ of (1.2), which also improves Theorem 3.1 in [4].

In [6], the following result was established:

Theorem 3.1. *Let $q_i=1/n$, then for any integer $2 \leq p \leq n$,*

$$\Big(\sum_{i=1}^n x_i\Big)^p \leq \Big(n^p-\lambda\binom{n}{p}\Big)M_{n,2}^p+\lambda E_{n,p},$$

with

$$\lambda=\frac{n^p(1-1/n)^{p/2}-(n-1)^p}{\binom{n}{p}(1-1/n)^{p/2}-\binom{n-1}{p}}.$$

We now use a method in [15] (see the proof of Theorem 2 there) to prove a result similar to the above. We note also that the same method can be applied to give another proof of Theorem 3.1 in [4] as well as (3.3) and we will leave the details to the interested reader.

Theorem 3.2. *For $r \geq 2$, $0 < p \leq n$,*

$$\Big(\sum_{i=1}^n x_i\Big)^p \leq (n-1)^{p(r-1)/r}\Big(\sum_{i=1}^n x_i^r\Big)^{p/r} + \Big(n^p - n^{p/r}(n-1)^{p(r-1)/r}\Big)\Big(\prod_{i=1}^n x_i\Big)^{p/n}. \tag{3.4}$$

For $1 < r \leq 2$, $p \geq n/(n-1)$,

$$\Big(\sum_{i=1}^n x_i\Big)^p \geq \Big(\sum_{i=1}^n x_i^r\Big)^{p/r} + \Big(n^p - n^{p/r}\Big)\Big(\prod_{i=1}^n x_i\Big)^{p/n}. \tag{3.5}$$

Proof. We first note that by a change of variables: $x_i \to x_i^{1/r}$, (3.4) is equivalent to (1.1) for the set $\{1, 1/r, 0\}$ with $\alpha = p/r$ and (3.5) is equivalent to (1.2) for the set $\{1, 1/r, 0\}$ with $\alpha = p/r$ Thus by Theorem 3.2 of [5], it suffices to prove (3.4) for the case $p = n$ and (3.5) for the case $p = n/(n-1)$.

We first prove (3.4) and we consider the quotient

$$f(\mathbf{x}) := \frac{\Big(\sum_{i=1}^n x_i\Big)^n - (n-1)^{n(r-1)/r}\Big(\sum_{i=1}^n x_i^r\Big)^{n/r}}{\prod_{i=1}^n x_i}.$$

As in the proof of Corollary 3.1, it suffices to consider the situation $0 < x = x_1 = \cdots = x_k < x_{k+1}$ for some $1 \leq k < n$ and to show that in this case $\partial f/\partial x > 0$. Without loss of generality, we assume from now on that $x_1 < x_2$. By setting

$$t = \Big(\frac{\sum_{i=2}^n x_i^r}{n-1}\Big)^{1/r},$$

we obtain

$$\begin{aligned}
& x_1\Big(\prod_{i=1}^n x_i\Big)\frac{\partial f}{(n-1)\partial x_1} \\
= \;& \Big(\sum_{i=1}^n x_i\Big)^{n-1}\Big(x_1 - \frac{\sum_{i=2}^n x_i}{n-1}\Big) - (n-1)^{n(r-1)/r}\Big(\sum_{i=1}^n x_i^r\Big)^{n/r-1}\Big(x_1^r - t^r\Big). \\
\geq \;& (n-1)^{n-1}\Bigg(\Big(\frac{x_1}{n-1} + \frac{\sum_{i=2}^n x_i}{n-1}\Big)^{n-1}(x_1 - t) - \Big(\frac{x_1^r}{n-1} + t^r\Big)^{n/r-1}\Big(x_1^r - t^r\Big)\Bigg) \\
\geq \;& (n-1)^{n-1}\Bigg(\Big(\frac{x_1}{n-1} + t\Big)^{n-1}(x_1 - t) - \Big(\frac{x_1^r}{n-1} + t^r\Big)^{n/r-1}\Big(x_1^r - t^r\Big)\Bigg) \\
= \;& (n-1)^{n-1}x_1^n\Bigg(\Big(\frac{1}{n-1} + \frac{t}{x_1}\Big)^{n-1}(1 - \frac{t}{x_1}) - \Big(\frac{1}{n-1} + (\frac{t}{x_1})^r\Big)^{n/r-1}\Big(1 - (\frac{t}{x_1})^r\Big)\Bigg).
\end{aligned}$$

We want to show the last expression above is positive and on setting $y = tx_1$, this is equivalent to showing that

$$g(y, n) = \frac{\Big(\frac{1}{n-1} + y\Big)^{n-1}(y-1)}{\Big(\frac{1}{n-1} + y^r\Big)^{n/r-1}(y^r - 1)} \leq 1,$$

for $y \geq 1$. Calculation yields that

$$\begin{aligned}
& \frac{\left(\left(\frac{1}{n-1}+y^r\right)^{n/r-1}(y^r-1)\right)^2}{\left(\frac{1}{n-1}+y\right)^{n-2}\left(\frac{1}{n-1}+y^r\right)^{n/r-2}}\frac{\partial g}{\partial y} \\
= & \big(ny+\frac{1}{n-1}-(n-1)\big)(y^r-1)(y^r+\frac{1}{n-1}) \\
& -(y+\frac{1}{n-1})(y^r-y^{r-1})\big(ny^r+\frac{r}{n-1}-(n-r)\big) \\
= & \frac{n}{n-1}\Big(y^{2r-1}+(1-r)y^{r+1}+\frac{n-2}{n-1}(r-1)y^r+(\frac{r}{n-1}-1)y^{r-1}-y+\frac{n-2}{n-1}\Big).
\end{aligned}$$

We now set $s=1/(n-1)$ so that $0 \leq s \leq 1$ and we define

$$a(y,s)=y^{2r-1}+(1-r)y^{r+1}+(1-s)(r-1)y^r+(rs-1)y^{r-1}-y+(1-s).$$

It then follows from Cauchy's mean value theorem that

$$\frac{\partial a}{\partial s}=-(r-1)y^r+ry^{r-1}-1=ry^{r-1}(1-y)-(1-y^r) \leq 0.$$

Thus

$$a(y,s) \geq a(y,1)=y(y^{2r-2}+(1-r)y^r+(r-1)y^{r-2}-1):=y\cdot b(y).$$

Now

$$b'(y)=(r-1)y^{r-3}(2y^r-ry^2+r-2):=(r-1)y^{r-3}c(y).$$

One checks easily that for $r \geq 2$, the function $c(y)$ is an increasing function of $y \geq 1$ and hence $c(y) \geq c(1)=0$ so that $b(y)$ is an increasing function of $y \geq 1$ and that $b(y) \geq b(1)=0$. This implies that $a(y,s) \geq 0$ so that $g(y,n)$ is an increasing function of y and we then deduce that

$$g(y,n) \leq \lim_{y \to +\infty} g(y,n)=1.$$

This shows that $\frac{\partial f}{\partial x_1} \geq 0$ and (3.4) now follows from our discussions above.

Now, to prove (3.5), we consider

$$h(\mathbf{x}):=\frac{\left(\sum_{i=1}^n x_i\right)^{n/(n-1)}-\left(\sum_{i=1}^n x_i^r\right)^{n/(n-1)r}}{\left(\prod_{i=1}^n x_i\right)^{1/(n-1)}}.$$

Similar to our discussion above, one can assume $x_{n-1}<x_n$ and it suffices to show $\partial h/\partial x_n>0$. Note that

$$\begin{aligned}
& x_n\Big(\prod_{i=1}^n x_i\Big)^{1/(n-1)}\frac{\partial h}{\partial x_n} \\
= & (n-1)^{1/(n-1)}\Big(\frac{x_n}{n-1}+\frac{\sum_{i=1}^{n-1}x_i}{n-1}\Big)^{1/(n-1)}\Big(x_n-\frac{\sum_{i=1}^{n-1}x_i}{n-1}\Big) \\
& -\Big(\sum_{i=1}^n x_i^r\Big)^{\frac{n}{(n-1)r}-1}\Big(x_n^r-\frac{\sum_{i=1}^{n-1}x_i^r}{n-1}\Big) \\
\geq & (n-1)^{1/(n-1)}\Big(\frac{x_n}{n-1}+\Big(\frac{\sum_{i=1}^{n-1}x_i}{n-1}\Big)^{1/r}\Big)^{1/(n-1)}\Big(x_n-\Big(\frac{\sum_{i=1}^{n-1}x_i}{n-1}\Big)^{1/r}\Big) \\
& -\Big(\sum_{i=1}^n x_i^r\Big)^{\frac{n}{(n-1)r}-1}\Big(x_n^r-\frac{\sum_{i=1}^{n-1}x_i^r}{n-1}\Big),
\end{aligned}$$

where the inequality follows from the observation that the function

$$z \mapsto \Big(\frac{x_n}{n-1}+z\Big)^{1/(n-1)}\Big(x_n - z\Big)$$

is decreasing for $0 < z < x_n$.

By proceeding similarly as in the proof of (3.4) above, one is then able to establish (3.5) and we shall omit all the details here. □

We point out here inequality (3.4) doesn't hold for $p > n$. To see this, we consider the case $x_1 = t, x_2 = \cdots = x_n = 1$, in which case (3.4) is reduced to

$$(n-1)^{p(r-1)/r}\Big(n-1+t^r\Big)^{p/r} + \Big(n^p - n^{p/r}(n-1)^{p(r-1)/r}\Big)t^{\frac{p}{n}} - (t+n-1)^p \geq 0.$$

We denote the left-hand side above as $f(t)$, then one checks easily that $f'(0) < 0$ for $p > n$ and $f(0) = 0$. This means $f(t) < 0$ for $t > 0$ sufficiently small and consequently (3.4) doesn't hold in this case. Similarly, the case $x_1 = x_2 = \cdots = x_{n-1} = t, x_n = 1$ shows that (3.5) doesn't hold for $p < n/(n-1)$.

4. Inequalities involving the Generalized Logarithmic Mean

In this section, the mean $M_{2,r}$ is always equipped with $q_1 = q_2 = 1/2$ and in this case we note that $L_2 = A_2$ and $L_{-1} = G_2$. Now we need a result of Pittenger [7]:

Theorem 4.1. *Let $a_1(r) = (r+1)/3$ and let $a_2(r) = (r-1)\log 2/\log r$ for $r > 0$ with $a_2(1) = \log 2$. For $r > 0$ let $b_1(r) = \min(a_1(r), a_2(r))$ and $b_2(r) = \max(a_1(r), a_2(r))$. For $r < 0$ let $b_1(r) = \min(0, a_1(r))$ and $b_2(r) = \max(0, a_1(r))$. Then for $x > 0, y > 0, x \neq y$,*

$$M_{2,b_1(r)} \leq L_r(x,y) \leq M_{2,b_2(r)},$$

with the choices $b_1(r), b_2(r)$ best possible.

We note here that Stolarsky [13] also showed $L_r(x,y) \leq M_{2,(r+1)/3}$ for $-1 \leq r \leq 1/2$ or $r \geq 2$ and $L_r(x,y) \geq M_{2,(r+1)/3}$ for $1/2 \leq r \leq 2$ or $r \leq -1$. Our next result gives an extension of Theorem 3.1 in [4] for in the case $n = 2, q_1 = q_2 = 1/2$. We leave the proof to the interested reader by pointing out that one can prove it by using similar approaches as in the proof of Theorem 2.1.

Theorem 4.2. *Let $n = 2, q_1 = q_2 = 1/2$. For $1/2 \leq s < 1$,*

$$C_{1,s,0}(\frac{1}{2}) \geq \Delta_{1,s,0} \geq \frac{1}{1-s}, \tag{4.1}$$

with the above inequality reversed when $0 < s \leq 1/2$. Moreover, for $r > 1$,

$$C_{r,1,0}(\frac{1}{2}) \geq \Delta_{r,1,0} \geq \frac{r}{r-1}.$$

We note here the lower bound in (4.1) (and the corresponding upper bound when $0 < s \leq 1/2$) is due to Seiffert [10], who also mentioned that the inequality $(G_2 + 2A_2)/3 \leq M_{2,2/3}$ was proposed at the "16th Austrian-Polish Mathematics Competition 1993". Stolarsky [13] has shown that $M_{2,2/3} < I$, it then follows that $(G_2 + 2A_2)/3 < I$, a result of Sándor [8]. One can obtain similar results by using Theorem 4.1 and 4.2 via this approach. Before we state our next result, we need a lemma:

Lemma 4.1. *For $t \geq 0, s \geq 1/2$,*

$$\Big(\frac{t^s-1}{t-1}\Big)^{1/(s-1)} \geq (1+t^s)^{1/s},$$

with the above inequality reversed when $0 < s \leq 1/2$.

Proof. We will prove the case $1/2 < s < 1$ and the proofs for the other cases are similar. By homogeneity we may also assume that $0 \leq t \leq 1$. Define

$$f(t) = \Big(\frac{t^s - 1}{t-1}\Big)^{1/(s-1)}(1+t^s)^{-1/s},$$

and it suffices to show $f(t) \geq 1$. Note that

$$f'(t) = \Big(\frac{t^s-1}{t-1}\Big)^{(2-s)/(s-1)}(1+t^s)^{-(1+s)/s}\Big(\frac{t^{2s-1}-(2s-1)t^s+(2s-1)t^{s-1}-1}{(1-s)(t-1)^2}\Big).$$

Let

$$g(t) = t^{2s-1} - (2s-1)t^s + (2s-1)t^{s-1} - 1.$$

Then

$$g'(t) = (2s-1)t^{s-2}(t^s - st + (s-1)) \leq 0.$$

Hence $g(t) \geq g(1) = 0$ and it follows $f'(t) \geq 0$ so that $f(t) \geq f(0) = 1$ which completes the proof. □

Corollary 4.1. *For $x, y > 0, x \neq y$,*

$$\frac{2}{e}L_2(x,y) \leq I(x,y) \leq \frac{2}{e}L_2(x,y) + (1-\frac{2}{e})L_{-1}(x,y).$$

Proof. The first inequality is due to Sándor [9] and it follows from Lemma 4.1 by replacing t with x/y and letting $s \to 1^-$. The second inequality is due to Alzer and Qiu [2, Theorem1], it follows from the inequality $I(x,y) \leq M_{2,\log 2}$ by Theorem 4.1 and the inequality $M_{2,\log 2} \leq \frac{2}{e}L_2(x,y) + (1-\frac{2}{e})L_{-1}(x,y)$ by Theorem 4.2 with $s = \log 2$. □

Corollary 4.2. *For $x > 0, y > 0, x \neq y$ and $0 < r \leq 1/2$,*

$$\Big(1 - \frac{2}{r^{1/(r-1)}}\Big)L_{-1}(x,y) + \frac{2}{r^{1/(r-1)}}L_2(x,y) \leq L_r(x,y) \leq \frac{(2-r)L_{-1}(x,y)+(1+r)L_2(x,y)}{3}, \tag{4.2}$$

with the above inequality reversed when $1/2 \leq r \leq 2, r \neq 1$.

We point out here since $r \mapsto L_r(x,y), x \neq y$ is strictly increasing (see [12]), one has $L_r \leq L_2 = A_2$ for $0 \leq r \leq 2$ and it follows from Theorem 4.1 that $(r-1)\log 2/\log r \leq 1$ for $1/2 \leq r \leq 2, r \neq 1$ (since $L_r \leq A_2$ and $L_r \leq M_{2,(r-1)\log 2/\log r}$ which is best possible). Hence one can apply Theorem 4.1 and inequality (4.1) and its counterpart for $0 \leq s \leq 1/2$ to prove Corollary 4.2.

Note here the limit case of $r \to 0^+$ for the right-hand side inequality of (4.2) gives a result of Carlson [3]. Similarly, the limit case of $r \to 1$ for the corresponding inequality gives a result of Sándor [8].

Acknowledgement

This work was partially carried out while the author was visiting the American Institute of Mathematics. We would like to thank both the American Institute of Mathematics and the Centre de Recherches Mathématiques at the Université de Montréal for their generous support and hospitality.

References

[1] H. Alzer, The inequality of Ky Fan and related results, *Acta Appl. Math.*, **38** (1995), 305-354.
[2] H. Alzer and S.-L. Qiu, Inequalities for means in two variables, *Arch. Math. (Basel)*, **80** (2003), 201–215.
[3] B. C. Carlson, The logarithmic mean, *Amer. Math. Monthly*, **79** (1972), 615–618.
[4] P. Gao, On an inequality of Diananda, *Int. J. Math. Math. Sci.*, **2003** (2003), 2061-2068.
[5] P. Gao, On an inequality of Diananda. II, *Int. J. Math. Math. Sci.*, **2005** (2005), 1381-1386.
[6] Z. Luo, L. Li and J.-J. Wen, The dimension-descending computation method of global optimizations for symmetric functions, *J. Chengdu Univ. Natur. Sci.*, **21** (2002), 5-11 (in Chinese).

[7] A. O. Pittenger, Inequalities between arithmetic and logarithmic means, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, no. 678-715 (1980), 15-18 (1981).
[8] J. Sándor, A note on some inequalities for means, *Arch. Math. (Basel)*, **56** (1991), 471-473.
[9] J. Sándor, On certain inequalities for means, *J. Math. Anal. Appl.*, **189** (1995), 602-606.
[10] H.-J. Seiffert, Solution of problem 10289, *Amer. Math. Monthly*, **103** (1996), 696-697.
[11] D.-W. Shen, On the coefficients and zeros of a polynomial, *J. Approx. Theory*, **96** (1999), 405-410.
[12] K. B. Stolarsky, Generalizations of the logarithmic mean, *Math. Mag.*, **48** (1975), 87-92.
[13] K. B. Stolarsky, The power and generalized logarithmic means, *Amer. Math. Monthly*, **87** (1980), 545-548.
[14] C. Wu, W. Wang and L. Fu, Inequalities for symmetric functions and their applications, *J. Chengdu Univ. Sci. Tech.*, no. 1 (1982), 103-108 (in Chinese).
[15] S.-H. Wu, Generalization and sharpness of the power means inequality and their applications, *J. Math. Anal. Appl.*, **312** (2005), 637-652.

Centre de recherches mathématiques, Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal (Québec), H3C 3J7, Canada
E-mail address: `gao@CRM.UMontreal.ca`