REVERSES OF THE TRIANGLE INEQUALITY VIA SELBERG'S AND BOAS-BELLMAN'S INEQUALITIES

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Abstract. Reverses of the triangle inequality for vectors in inner product spaces via the Selberg and Boas-Bellman generalisations of Bessel's inequality are given. Applications for complex numbers are also provided.

1. Introduction

In 1966, J.B. Diaz and F.T. Metcalf [3] obtained the following reverse of the triangle inequality on utilising an argument based on the Bessel inequality in a real or complex inner product space $(H, \langle \cdot, \cdot \rangle)$.

Theorem 1 (Diaz-Metcalf, 1966)**.** *Let $e_1, \dots, e_m$ be orthonormal vectors in H, i.e., $e_i \perp e_j$ for $i \neq j$ and $\|e_i\| = 1$, $i, j \in \{1, \dots, m\}$. Suppose the vectors $x_1, \dots, x_n \in H \setminus \{0\}$ satisfy*

$$0 \leq r_k \leq \frac{\operatorname{Re} \langle x_j, e_k \rangle}{\|x_j\|}, \quad j \in \{1, \dots, n\}, \ k \in \{1, \dots, m\}. \tag{1.1}$$

Then

$$\left(\sum_{k=1}^{n} r_k^2 \right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\|, \tag{1.2}$$

where the equality holds if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} r_k e_k. \tag{1.3}$$

In an attempt to improve this result for the case of complex inner product spaces, the author obtained in 2004 the following result [4]:

Theorem 2 (Dragomir, 2004)**.** *Let $e_1, \dots, e_m \in H$ be an orthonormal family of vectors in the complex inner product space H. If the vectors $x_1, \dots, x_n \in H$ satisfy the conditions*

$$0 \leq r_k \|x_j\| \leq \operatorname{Re} \langle x_j, e_k \rangle, \quad 0 \leq \rho_k \|x_j\| \leq \operatorname{Im} \langle x_j, e_k \rangle \tag{1.4}$$

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for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then we have the following reverse of the generalised triangle inequality:*

$$\left[\sum_{k=1}^{m} \left(r_k^2 + \rho_k^2\right)\right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{1.5}$$

The equality holds in (1.5) if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} \left(r_k + i\rho_k\right) e_k. \tag{1.6}$$

As particular cases of interest, we can notice the following results [4]:

Corollary 1. *Let* $e_1, \ldots, e_m \in H$ *be orthonormal vectors in the complex inner product space* $(H; \langle\cdot,\cdot\rangle)$ *and* $\rho_k, \eta_k \in (0,1)$, $k \in \{1, \ldots, m\}$. *If* $x_1, \ldots, x_k \in H$ *are such that*

$$\|x_j - e_k\| \leq \rho_k, \quad \|x_j - ie_k\| \leq \eta_k \tag{1.7}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then we have the inequality*

$$\left[\sum_{k=1}^{m} \left(2 - \rho_k^2 - \eta_k^2\right)\right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{1.8}$$

The case of equality holds in (1.8) if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} \left(\sqrt{1-\rho_k^2} + i\sqrt{1-\eta_k^2}\right) e_k. \tag{1.9}$$

Corollary 2. *Let* $e_1, \ldots, e_m$ *be as in Corollary 1 and* $M_k \geq m_k > 0$, $N_k \geq n_k > 0$, $k \in \{1, \ldots, m\}$. *If* $x_1, \ldots, x_n \in H$ *are such that either*

$$\operatorname{Re}\langle M_k e_k - x_j, x_j - m_k e_k\rangle \geq 0, \quad \operatorname{Re}\langle N_k e_k - x_j, x_j - n_k e_k\rangle \geq 0 \tag{1.10}$$

or, equivalently,

$$\left\|x_j - \frac{M_k + m_k}{2} e_k\right\| \leq \frac{1}{2}\left(M_k - m_k\right), \tag{1.11}$$
$$\left\|x_j - \frac{N_k + n_k}{2} e_k\right\| \leq \frac{1}{2}\left(N_k - n_k\right),$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then we have the inequality*

$$2\left\{\sum_{k=1}^{m} \left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2}\right]\right\}^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{1.12}$$

The case of equality holds in (1.12) if and only if

$$\sum_{j=1}^{n} x_j = 2\left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} \left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + i\frac{\sqrt{n_k N_k}}{N_k + n_k}\right) e_k.$$

In the above results the vectors $\{e_1, \ldots, e_m\}$ are assumed to be orthonormal and the principle tool in proving these results is the well known Bessel's inequality:

$$\sum_{k=1}^{m} |\langle x, e_k \rangle|^2 \leq \|x\|^2, \quad x \in H. \tag{1.13}$$

If we use the following generalisation of Bessel's inequality, namely:

$$\sum_{k=1}^{m} \frac{|\langle x, y_k \rangle|^2}{\sum_{j=1}^{m} |\langle y_k, y_j \rangle|} \leq \|x\|^2, \tag{1.14}$$

provided $x, y_1, \ldots, y_m$ are vectors in H and $y_k \neq 0$, $k \in \{1, \ldots, m\}$, which is known in the literature as the *Selberg inequality,* (see [6, p. 394] of [5, p. 134]), then we can obtain different reverses of the generalised triangle inequality, where the assumption of orthonormality is taken out.

A similar approach may be considered if the other generalisation of Bessel's inequality due to Boas [2] and Bellman [1] is used, namely the inequality

$$\sum_{k=1}^{m} |\langle x, y_k \rangle|^2 \leq \|x\|^2 \left[\max_{1 \leq k \leq m} \|y_k\|^2 + \left(\sum_{1 \leq k \neq j \leq m} |\langle y_k, y_j \rangle|^2 \right)^{1/2} \right], \tag{1.15}$$

where $x, y_1, \ldots, y_m$ are as above.

The main aim of this paper is to establish new reverses of the triangle inequality for $x_1, \ldots, x_n$ vectors in an inner product space H in terms of another sequence $y_1, \ldots, y_m$ of nonzero vectors that can be non-orthonormal. The main tools in obtaining such results are the Selberg inequality (1.14) and the Boas-Bellman inequality (1.15). Applications for complex numbers are also given.

2. The Results

The following result holds:

Theorem 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be a complex inner product space, $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ be vectors such that there exist the nonnegative real numbers ρ_j, η_j, $j \in \{1, \ldots, m\}$ with*

$$\operatorname{Re} \langle x_i, y_j \rangle \geq \rho_j \|x_i\| \|y_j\|, \quad \operatorname{Im} \langle x_i, y_j \rangle \geq \eta_j \|x_i\| \|y_j\|, \tag{2.1}$$

for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then

$$\left\| \sum_{i=1}^{n} x_i \right\| \geq \left(\sum_{j=1}^{m} \frac{(\rho_j^2 + \eta_j^2) \|y_j\|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|} \right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\|. \tag{2.2}$$

Proof. Utilising Selberg's inequality, we have

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 \geq \sum_{j=1}^{m} \frac{|\langle \sum_{i=1}^{n} x_i, y_j \rangle|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|}. \tag{2.3}$$

Since

$$\left| \left\langle \sum_{i=1}^{n} x_i, y_j \right\rangle \right|^2 = \left(\sum_{i=1}^{n} \operatorname{Re} \langle x_i, y_j \rangle \right)^2 + \left(\sum_{i=1}^{n} \operatorname{Im} \langle x_i, y_j \rangle \right)^2,$$

then, by (2.1), we obtain

$$\left|\left\langle\sum_{i=1}^{n}x_i,y_j\right\rangle\right|^2 \geq \rho_j^2\left\|y_j\right\|^2\left(\sum_{i=1}^{n}\|x_i\|\right)^2+\eta_j^2\left\|y_j\right\|^2\left(\sum_{i=1}^{n}\|x_i\|\right)^2 = \left(\sum_{i=1}^{n}\|x_i\|\right)^2\left(\rho_j^2+\eta_j^2\right)\left\|y_j\right\|^2, \tag{2.4}$$

for any $j\in\{1,\dots,m\}$. Therefore, by (2.3) we get

$$\left\|\sum_{i=1}^{n}x_i\right\|^2\geq\left(\sum_{i=1}^{n}\|x_i\|\right)^2\sum_{j=1}^{m}\frac{\left(\rho_j^2+\eta_j^2\right)\left\|y_j\right\|^2}{\sum_{k=1}^{m}\left|\langle y_j,y_k\rangle\right|},$$

which is clearly equivalent to (2.2). ∎

Remark 1. *If the space is real or complex and only the first condition of (2.1) is available, then*

$$\left\|\sum_{i=1}^{n}x_i\right\|\geq\left(\sum_{j=1}^{m}\frac{\rho_j^2\left\|y_j\right\|^2}{\sum_{k=1}^{m}\left|\langle y_j,y_k\rangle\right|}\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\|. \tag{2.5}$$

Remark 2. *If $\{y_1,\dots,y_m\}$ are orthonormal and*

$$\operatorname{Re}\langle x_i,y_j\rangle\geq\rho_j\|x_i\|,\quad \operatorname{Im}\langle x_i,y_j\rangle\geq\eta_j\|x_i\|, \tag{2.6}$$

for $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$, then

$$\left\|\sum_{i=1}^{n}x_i\right\|\geq\left(\sum_{j=1}^{m}\left(\rho_j^2+\eta_j^2\right)\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| \tag{2.7}$$

and the inequality (2.5) is recaptured.

The following corollary may be of interest for applications:

Corollary 3. *Let $y_1,\dots,y_m$ be nonzero vectors in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$ and $p_k,q_k\in(0,1)$ for $k\in\{1,\dots,m\}$. If $x_1,\dots,x_n\in H$ are such that:*

$$\|x_j-y_k\|\leq p_k<\|y_k\|,\quad \|x_j-iy_k\|\leq q_k<\|y_k\| \tag{2.8}$$

for each $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$, then:

$$\left\|\sum_{j=1}^{n}x_j\right\|\geq\left(\sum_{k=1}^{m}\frac{2\left\|y_k\right\|^2-p_k^2-q_k^2}{\sum_{s=1}^{m}\left|\langle y_k,y_s\rangle\right|}\right)^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\|. \tag{2.9}$$

Proof. From the first inequality in (2.8) we deduce, by taking the square, that

$$\|x_j\|^2+\|y_k\|^2-p_k^2\leq 2\operatorname{Re}\langle x_j,y_k\rangle,$$

implying

$$\frac{\|x_j\|^2}{\sqrt{\|y_k\|^2-p_k^2}}+\sqrt{\|y_k\|^2-p_k^2}\leq\frac{2\operatorname{Re}\langle x_j,y_k\rangle}{\sqrt{\|y_k\|^2-p_k^2}} \tag{2.10}$$

since $\sqrt{\|y_k\|^2-p_k^2}>0$ for $j\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$.

On the other hand, obviously,

$$2\|x_j\| \le \frac{\|x_j\|^2}{\sqrt{\|y_k\|^2 - p_k^2}} + \sqrt{\|y_k\|^2 - p_k^2}, \tag{2.11}$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Hence, by (2.10) and (2.11) we have

$$\|x_j\| \sqrt{\|y_k\|^2 - p_k^2} \le \operatorname{Re}\langle x_j, y_k\rangle. \tag{2.12}$$

Since $\operatorname{Re}\langle x_j, iy_k\rangle = \operatorname{Im}\langle ix_j, y_k\rangle$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ then, by the second inequality in (2.8), we have

$$\|x_j\| \sqrt{\|y_k\|^2 - q_k^2} \le \operatorname{Im}\langle x_j, y_k\rangle, \tag{2.13}$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Now if we define

$$\rho_k = \frac{\sqrt{\|y_k\|^2 - p_k^2}}{\|y_k\|}, \quad \eta_k = \frac{\sqrt{\|y_k\|^2 - q_k^2}}{\|y_k\|}, \quad k \in \{1, \ldots, m\}$$

and apply Theorem 3, we get

$$\left\|\sum_{j=1}^n x_j\right\| \ge \left(\sum_{k=1}^m \frac{\left(\frac{\|y_k\|^2 - p_k^2}{\|y_k\|^2} + \frac{\|y_k\|^2 - q_k^2}{\|y_k\|^2}\right) \cdot \|y_k\|^2}{\sum_{s=1}^m |\langle y_k, y_s\rangle|}\right)^{\frac{1}{2}} \sum_{j=1}^n \|x_j\|,$$

which is exactly (2.9). ∎

Remark 3. *If $\{y_1, \ldots, y_m\}$ are orthonormal and $p_k, q_k \in (0,1)$, then out of (2.9) we can recapture (1.8) from the introduction.*

The following corollary may be stated as well:

Corollary 4. *Let $y_1, \ldots, y_m$ be nonzero vectors in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $M_k \ge m_k > 0$, $N_k \ge n_k > 0$ for $k \in \{1, \ldots, m\}$. If $x_1, \ldots, x_n \in H$ are such that:*

$$\begin{aligned} &\operatorname{Re}\langle M_k y_k - x_j, x_j - m_k y_k\rangle \ge 0 \quad \textit{and} \\ &\operatorname{Re}\langle N_k i y_k - x_j, x_j - n_k i y_k\rangle \ge 0 \end{aligned} \tag{2.14}$$

or, equivalently,

$$\begin{aligned} &\left\| x_j - \frac{M_k + m_k}{2} y_k \right\| \le \frac{1}{2}(M_k - m_k)\|y_k\| \quad \textit{and} \\ &\left\| x_j - \frac{N_k + n_k}{2} i y_k \right\| \le \frac{1}{2}(N_k - n_k)\|y_k\| \end{aligned} \tag{2.15}$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$\left\|\sum_{j=1}^n x_j\right\| \ge 2 \cdot \left[\sum_{k=1}^m \left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2}\right] \times \frac{\|y_k\|^2}{\sum_{s=1}^m |\langle y_k, y_s\rangle|}\right]^{\frac{1}{2}} \sum_{j=1}^n \|x_j\|. \tag{2.16}$$

Proof. From the first inequality in (2.14) we get

$$\|x_j\|^2 + M_k m_k \|y_k\|^2 \leq (M_k + m_k) \operatorname{Re} \langle x_j, y_k \rangle ,$$

implying

$$\frac{\|x_j\|^2}{\sqrt{M_k m_k}} + \sqrt{M_k m_k} \|y_k\|^2 \leq \frac{M_k + m_k}{\sqrt{M_k m_k}} \operatorname{Re} \langle x_j, y_k \rangle \tag{2.17}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Since, obviously

$$2 \|x_j\| \|y_k\| \leq \frac{\|x_j\|^2}{\sqrt{M_k m_k}} + \sqrt{M_k m_k} \|y_k\|^2 , \tag{2.18}$$

then, by (2.17) and (2.18), we get

$$\frac{2\sqrt{M_k m_k}}{M_k + m_k} \|x_j\| \|y_k\| \leq \operatorname{Re} \langle x_j, y_k \rangle \tag{2.19}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

In a similar manner, on utilising the second part of (2.14) we get

$$\frac{2\sqrt{n_k N_k}}{N_k + n_k} \|x_j\| \|y_k\| \leq \operatorname{Im} \langle x_j, y_k \rangle \tag{2.20}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Applying Theorem 3 for

$$\rho_k = \frac{2\sqrt{M_k m_k}}{M_k + m_k}, \quad \eta_k = \frac{2\sqrt{n_k N_k}}{N_k + n_k}, \quad k \in \{1, \ldots, m\}$$

we deduce the desired result (2.16). ∎

Remark 4. *The case when $\{y_1, \ldots, y_m\}$ becomes an orthonormal family of vectors will provide the known inequality (1.12) of the introduction.*

On utilising the other generalisation of Bessel's inequality we can provide the following reverse of the triangle inequality as well:

Theorem 4. *With the assumptions of Theorem 3 for the vectors $x_1, \ldots, x_n, y_1, \ldots, y_m$ and the nonnegative real numbers $\rho_j, \eta_j,\ j \in \{1, \ldots, m\}$, we have the inequality*

$$\left\| \sum_{i=1}^{n} x_i \right\| \geq \left(\frac{\sum_{j=1}^{m} \left(\rho_j^2 + \eta_j^2 \right) \|y_j\|^2}{\max_{1 \leq i \leq m} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \rangle|^2 \right)^{1/2}} \right)^{1/2} \sum_{i=1}^{n} \|x_i\| . \tag{2.21}$$

Proof. The argument is similar with the one incorporated in the proof of Theorem 3 by utilising the inequality

$$\sum_{k=1}^{m} \left| \left\langle \sum_{i=1}^{n} x_i, y_k \right\rangle \right|^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 \left[\max_{1 \leq i \leq m} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \rangle|^2 \right)^{1/2} \right]$$

that follows from (1.15). ∎

Remark 5. *Similar results with those incorporated in Corolaries 3-4 may be stated as well. The details are omitted.*

Remark 6. *If one utilises the other generalisations of Bessel's inequalities as provided, for instance, in the monograph* [5], *Chapter 4, that one can state other reverses of the traingle inequality.*

3. Applications for Complex Numbers

The above results may be used in establishing some interesting reverses of the generalised triangle inequality for complex numbers.

Proposition 1. *Let* $c_1, \dots, c_n$ *and* $d_1, \dots, d_m$ *be complex numbers with the property that there exists the nonnegative real numbers* ρ_k, η_k, $k \in \{1, \dots, m\}$ *with*

$$\operatorname{Re} c_j \cdot \operatorname{Re} d_k + \operatorname{Im} c_j \cdot \operatorname{Im} d_k \geq \rho_k |c_j| |d_k| \tag{3.1}$$

and

$$\operatorname{Re} d_k \cdot \operatorname{Im} c_j - \operatorname{Re} c_j \cdot \operatorname{Im} d_k \geq \eta_k |c_j| |d_k|, \tag{3.2}$$

for any $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$. *Then*

$$\left| \sum_{j=1}^{n} c_j \right| \geq \left[\frac{\sum_{k=1}^{m} \left(\rho_k^2 + \eta_k^2 \right) |d_k|}{\sum_{s=1}^{m} |d_s|} \right]^{\frac{1}{2}} \sum_{j=1}^{n} |c_j|. \tag{3.3}$$

Proof. It follows from Theorem 3 applied for the complex inner product space $\mathbb{H} = \mathbb{C}$ endowed with the canonical inner product $\langle x, y \rangle := x\bar{y}$. The details are omitted. ∎

Possibly a more useful result which also has a clear geometrical interpretation is incorporated in the following.

Proposition 2. *Assume that the complex numbers* c_j, $d_j, j \in \{1, \dots, n\}$ *and the nonnegative real numbers* p_k, q_k $k \in \{1, \dots, m\}$ *are such that*

$$|c_j - d_k| \leq p_k < |d_k|; \quad |c_j - i d_k| \leq q_k < |d_k| \tag{3.4}$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$. *Then*

$$\left| \sum_{j=1}^{n} c_j \right| \geq \frac{\left(\sum_{k=1}^{m} \frac{2|d_k|^2 - p_k^2 - q_k^2}{|d_k|} \right)^{\frac{1}{2}}}{\left(\sum_{s=1}^{m} |d_s| \right)^{\frac{1}{2}}} \sum_{j=1}^{n} |c_j|. \tag{3.5}$$

The proof is obvious by Corollary 3.

Further, on using Corollary 4 we can state:

Proposition 3. *If* $c_1, \dots, c_n$ *and* $d_1, \dots, d_m$ *are complex numbers such that there exists* $M_k \geq m_k > 0$, $N_k \geq n_k > 0$ *with*

$$\begin{aligned} &(M_k \operatorname{Re} d_k - \operatorname{Re} c_j)(\operatorname{Re} c_j - m_k \operatorname{Re} d_k) \\ &\qquad + (M_k \operatorname{Im} d_k - \operatorname{Im} c_j)(\operatorname{Im} c_j - m_k \operatorname{Im} d_k) \geq 0 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &(-N_k \operatorname{Im} d_k - \operatorname{Re} c_j)(\operatorname{Re} c_j + n_k \operatorname{Im} d_k) \\ &\qquad + (N_k \operatorname{Re} d_k - \operatorname{Im} c_j)(\operatorname{Im} c_j - n_k \operatorname{Re} d_k) \geq 0 \end{aligned} \tag{3.7}$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$\left|\sum_{j=1}^{n} c_j\right| \geq 2 \cdot \frac{\left(\sum_{k=1}^{m}\left[\frac{m_k M_k}{(M_k+m_k)^2} + \frac{n_k N_k}{(N_k+n_k)^2}\right]|d_k|\right)^{\frac{1}{2}}}{\left(\sum_{s=1}^{m}|d_s|\right)^{\frac{1}{2}}} \sum_{j=1}^{n}|c_j|. \tag{3.8}$$

Remark 7. *A sufficient condition for (3.6) to occur is*

$$m_k \operatorname{Re} d_k \leq \operatorname{Re} c_j \leq M_k \operatorname{Re} d_k$$

and

$$m_k \operatorname{Im} d_k \leq \operatorname{Im} c_j \leq M_k \operatorname{Im} d_k$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$ *while for (3.7) is*

$$-N_k \operatorname{Im} d_k \geq \operatorname{Re} c_j \geq -n_k \operatorname{Im} d_k$$

and

$$N_k \operatorname{Re} d_k \geq \operatorname{Im} c_j \geq n_k \operatorname{Re} d_k$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$.

Finally, on utilising Theorem 4 we can state the following reverse of the triangle inequality for complex numbers as well:

Proposition 4. *With the assumptions of Proposition 3 for the complex numbers* $c_1, \dots, c_n$; $d_1, \dots, d_m$ *and the nonnegative real numbers* ρ_k, η_k, $k \in \{1, \dots, m\}$, *we have*

$$\left|\sum_{j=1}^{n} c_j\right| \geq \left(\frac{\sum_{k=1}^{m}\left(\rho_k^2+\eta_k^2\right)\|d_k\|^2}{\max_{1\leq j\leq m}\|d_j\|^2 + \left(\sum_{1\leq j\neq k\leq m}|\langle d_j, d_k\rangle|^2\right)^{1/2}}\right)^{1/2} \sum_{j=1}^{n}|c_j|.$$

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