On certain inequalities for the Gamma function

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Abstract. We prove an extension, in an improved form, of a double inequality for the Gamma function.

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1 Introduction

Let Γ be the Euler gamma function, defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

In 1987, J. Lew, J. Frauenthal and N. Keyfitz [2], by studying certain problems of **traffic flow**, have proved the double inequality

$$2\Gamma\left(n+\frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(n+1) \leq 2^n\Gamma\left(n+\frac{1}{2}\right), \tag{1}$$

where $n \geq 1$ is a positive integer. See also M. J. Cloud and B. C. Drachman ([1], p.123).

We note that, since

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(n) = (n-1)!$$

and

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,$$

(see e.g. [3]) inequality (1) – having application to traffic flow, reduces to

$$2 \leq \frac{2^n n!}{(2n-1)!} \leq 2^n, \tag{2}$$

which can be established e.g. by mathematical induction.

Our aim in what follows, is to extend (1) for real arguments. In fact, a stronger relation will be obtained.

2 Main results

First we prove the following

Theorem 1. *For any $x > 0$ one has*

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} \leq \sqrt{x + \frac{1}{2}}. \tag{3}$$

For the proof of (3) we need the following auxiliary result due to J. Wendel [4]:

Lemma 1. *For all $0 < a < 1$ and all $x > 0$ one has*

$$\left(\frac{x}{x+a}\right)^{1-a} \leq \frac{\Gamma(x+a)}{x^a \Gamma(x)} \leq 1. \tag{4}$$

Proof. Apply the Hölder inequality

$$\int_0^\infty f(t)g(t)dt \le \left(\int_0^\infty (f(t))^p dt\right)^{1/p} \left(\int_0^\infty (g(t))^q\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. Put

$$p=\frac{1}{a},\quad q=\frac{1}{1-a},\quad f(t)=e^{-t}t^{x+a-1},\quad g(t)=e^{-t}t^{x-1}.$$

By $\Gamma(x+1)=x\Gamma(x)$, we obtain the right side of (4). The left side of (4) follows by putting $a\to 1-a$, $x\to x+a$ in the right side of (4).

Apply now Lemma 1 for $a=\frac{1}{2}$ in order to deduce

$$\frac{1}{\sqrt{x}} \le \frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)} \le \frac{1}{x}\sqrt{x+\frac{1}{2}}. \tag{5}$$

By multiplying both sides of (5) with $\sqrt{x}$ and using $x\Gamma(x)=\Gamma(x+1)$, (3) follows.

Lemma 2. *For any $x\ge 3/2$ one has*

$$\frac{2}{\sqrt{\pi}} < \sqrt{x} \text{ and } 4^x > \pi\left(x+\frac{1}{2}\right). \tag{6}$$

Proof. Since $\frac{2}{\sqrt{\pi}} < \sqrt{\frac{3}{2}} \le \sqrt{x}$, the first relation of (6) is trivial. By $\pi<4$, for the second part of (6) it will be sufficient to prove that

$$4^{x-1} \ge x+\frac{1}{2}. \tag{7}$$

Let $k(x)=4^{x-1}-x-\frac{1}{2}$, $x\ge\frac{3}{2}$. Since $k'(x)=4^{x-1}\ln 4-1\ge 2\ln 4-1>0$, k is strictly increasing, so $k(x)\ge k\left(\frac{3}{2}\right)=0$, with equality only for $x=\frac{3}{2}$.

Theorem 2. *For all* $x \geq \dfrac{3}{2}$ *one has*

$$\frac{2}{\sqrt{\pi}} < \sqrt{x} \leq \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} \leq \sqrt{x+\frac{1}{2}} < \frac{2^x}{\sqrt{\pi}}. \tag{8}$$

Proof. This follows by Theorem 1 and Lemma 2.

Remark. The weaker inequalities in (8) for $n \geq 2$ coincide with relation (1).

References

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