TWO NEW INEQUALITIES WITH RESPECT TO MEANS

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Abstract. By the properties of geometrically convex functions, this paper presents two new inequalities with respect to means.

1. Introduction

Throughout the paper we assume R^n be the n-dimensional Euclidean Space, $R_+^n = \{(x_1, x_2, \cdots, x_n), x_i > 0, i = 1, 2, \cdots, n\}$, and $x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha)$, $x \cdot y = (x_1y_1, x_2y_2, \cdots, x_ny_n)$, where $\alpha \in R$, and $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$.

Suppose $a_i > 0\,(i = 1, 2, \cdots, n)$, $\lambda_i > 0\,(i = 1, 2, \cdots, n)$, $\sum_{i=1}^n \lambda_i = 1$, and $\alpha > 0, \alpha \neq 1$. They are defined respectively by

$$A(a) = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad G(a) = \sqrt[n]{a_1a_2\cdots a_n},$$

$$A(a^\alpha) = \frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}, \quad G(a^\alpha) = \sqrt[n]{a_1^\alpha a_2^\alpha \cdots a_n^\alpha},$$

$$A(a, \lambda) = \lambda_1a_1 + \lambda_2a_2 + \cdots + \lambda_na_n, \quad G(a, \lambda) = a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n},$$

$$Q_n^{[k]} = \left[\prod_{1\le i_1<\cdots<i_k\le n}\left(\frac{a_{i_1} + a_{i_2} + \cdots + a_{i_k}}{k}\right)\right]^{\frac{1}{\binom{n}{k}}}, k = 1, 2, \cdots, n,$$

and $\bar{G} = (G(a), G(a), \cdots, G(a))$.

We shall prove

Theorem 1.1. *Let* $m = \min\{a_i; 1 \le i \le n\}$, $M = \max\{a_i; 1 \le i \le n\}$, *and*

$$K_\alpha = \begin{cases} \alpha^2M^{\alpha-1}, 0 < \alpha < 1 \\ \alpha^2m^{\alpha-1}, \alpha > 1 \end{cases}, L_\alpha = \begin{cases} \alpha^2m^{\alpha-1}, 0 < \alpha < 1 \\ \alpha^2M^{\alpha-1}, \alpha > 1 \end{cases}.$$

Then

$$K_\alpha[A(a) - G(a)] \le A(a^\alpha) - G(a^\alpha) \le L_\alpha[A(a) - G(a)]. \tag{1.1}$$

$$K_\alpha[A(a,\lambda) - G(a,\lambda)] \le A(a^\alpha,\lambda) - G(a^\alpha,\lambda) \le L_\alpha[A(a,\lambda) - G(a,\lambda)]. \tag{1.2}$$

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Theorem 1.2. *Let* $f : [a,b] \to (0,+\infty)$ *be continuous,* $\max_{x\in[a,b]} f(x) = M$, $\min_{x\in[a,b]} f(x) = m$, $p : [a,b] \to [0,+\infty)$ *be continuous,* $\int_a^b p(x)dx = 1$, *and*

$$K_\alpha = \begin{cases} \alpha^2 M^{\alpha-1}, 0<\alpha<1 \\ \alpha^2 m^{\alpha-1}, \alpha>1 \end{cases}, L_\alpha = \begin{cases} \alpha^2 m^{\alpha-1}, 0<\alpha<1 \\ \alpha^2 M^{\alpha-1}, \alpha>1 \end{cases},$$

$$A(f,p) = \int_a^b p(x) f(x)\, dx, \quad G(f,p) = \exp\left[\int_a^b p(x) \ln f(x)\, dx\right].$$

Then

$$K_\alpha [A(f,p) - G(f,p)] \le A(f^\alpha, p) - G(f^\alpha, p) \le L_\alpha [A(f,p) - G(f,p)]. \tag{1.3}$$

Theorem 1.3. *Let* $k = 1, 2, \cdots, n$, $\lambda = \frac{k(k-1)}{n(n-1)}$. *Then*

$$Q_n^{[k]}(a) \ge A^\lambda(a) \cdot G^{1-\lambda}(a). \tag{1.4}$$

2. Lemmas

Definition 2.1. ([1]) $H \subseteq R_+^n$ *is called geometrically convex set, if* $x^\beta y^{1-\beta} \in H$ *for any* $x, y \in H$, $0 < \beta < 1$.

Definition 2.2. ([1, 3]) *Let* $H \subseteq R_+^n$ *be a geometrically convex set,* $f : H \to (0,+\infty)$ *is continuous function.* f *is called geometrically convex function, if the following inequality*

$$f\left(x^\beta y^{1-\beta}\right) \le f^\beta(x) f^{1-\beta}(y) \tag{2.1}$$

holds for all $x, y \in H$ *and* $0 < \beta < 1$. *And* f *is called geometrically concave function, if the Inequality* (2.1) *is reversed.*

Definition 2.3. ([1, 2, 3]) *Let* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$,$y = (y_1, y_2, \cdots, y_n) \in R_+^n$, $\left(x_{[1]}, x_{[2]}, \cdots, x_{[n]}\right)$ *and* $\left(y_{[1]}, y_{[2]}, \cdots, y_{[n]}\right)$ *be the decreasing queue of* $(x_1, x_2, \cdots, x_n)$ *and* $(y_1, y_2, \cdots, y_n)$ *respectively. We say* $(x_1, x_2, \cdots, x_n)$ *logarithm majorizes* $(y_1, y_2, \cdots, y_n)$, *denotes* $\ln x \succ \ln y$, *if*

$$\begin{cases} \prod_{i=1}^k x_{[i]} \ge \prod_{i=1}^k y_{[i]}, k = 1, 2, \cdots, n-1, \\ x_{[1]} x_{[2]} \cdots x_{[n]} = y_{[1]} y_{[2]} \cdots y_{[n]}. \end{cases}$$

Lemma 2.1. ([1]) $a = (a_1, a_2, \cdots, a_n)$ *logarithm majorizes* $\bar{G} = (G(a), G(a), \cdots, G(a))$.

Definition 2.4. *Suppose* $E \subseteq R_+^n$, $f : E \to [0,+\infty)$. *Then* f *is called S-geometrically convex function, if for any* $x, y \in E \subseteq R_+^n$, *when* $\ln x \succ \ln y$, *have*

$$f(x) \ge f(y). \tag{2.2}$$

And f *is called S-geometrically concave function, if the Inequality* (2.2) *is reversed.*

Lemma 2.2. ([1]) *Let* $E \subseteq R_+^n$ *be symmetric logarithm convex set and have a nonempty interior,* $f : E \to [0,+\infty)$ *be symmetric continuously differentiable on the interior of* E *and continuous on* E. *Then* f *is a S-geometrically convex function, if the following inequality*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \ge 0 \tag{2.3}$$

holds for all x *in the interior of* E. *And* f *is a S-geometrically concave function, if the Inequality* (2.3) *is reversed.*

3. The proof of the main results

The proof of Theorem 1.1 Let $I = [m, M], E = I^n$, so E is a bound geometrically convex set, then exists constant C which makes the following inequality

$$f(a) = [A(a^\alpha) - G(a^\alpha)] - K_\alpha [A(a) - G(a)] + C \geq 0$$

holds for any $a \in E$. And f is a symmetric continuous and differentiable function on E,

$$\frac{\partial f}{\partial a_1} = \frac{\alpha}{n}\left[a_1^{\alpha-1} - \frac{G(a^\alpha)}{a_1}\right] - \frac{K_\alpha}{n}\left[1 - \frac{G(a)}{a_1}\right],$$

$$\frac{\partial f}{\partial a_2} = \frac{\alpha}{n}\left[a_2^{\alpha-1} - \frac{G(a^\alpha)}{a_2}\right] - \frac{K_\alpha}{n}\left[1 - \frac{G(a)}{a_2}\right],$$

$$(\ln a_1 - \ln a_2)\left(a_1\frac{\partial f}{\partial a_1} - a_2\frac{\partial f}{\partial a_2}\right) = \frac{1}{n}(\ln a_1 - \ln a_2)\left[\alpha\left(a_1^\alpha - a_2^\alpha\right) - K_\alpha (a_1 - a_2)\right]$$

$$= \frac{1}{n}(a_1 - a_2)(\ln a_1 - \ln a_2)\left[\alpha \cdot \frac{a_1^\alpha - a_2^\alpha}{a_1 - a_2} - K_\alpha\right].$$

From differential mean value theorem, there exists $\xi \in [a_1, a_2] \subseteq [m, M]$, or $\xi \in [a_2, a_1] \subseteq [m, M]$, has $\frac{a_1^\alpha - a_2^\alpha}{a_1 - a_2} = \alpha\xi^{\alpha-1}$, so

$$(\ln a_1 - \ln a_2)\left(a_1\frac{\partial f}{\partial a_1} - a_2\frac{\partial f}{\partial a_2}\right) = \frac{1}{n}(a_1 - a_2)(\ln a_1 - \ln a_2)\left[\alpha^2\xi^{\alpha-1} - K_\alpha\right].$$

If $\alpha > 1$, then

$$K_\alpha = \alpha^2 m^{\alpha-1} \leq \alpha^2\xi^{\alpha-1}, \quad (\ln a_1 - \ln a_2)\left(a_1\frac{\partial f}{\partial a_1} - a_2\frac{\partial f}{\partial a_2}\right) \geq 0.$$

If $0 < \alpha < 1$, then

$$K_\alpha = \alpha^2 M^{\alpha-1} \leq \alpha^2\xi^{\alpha-1}, \quad (\ln a_1 - \ln a_2)\left(a_1\frac{\partial f}{\partial a_1} - a_2\frac{\partial f}{\partial a_2}\right) \geq 0.$$

So f is a S-geometrically convex function with Lemma 2.2. From Lemma 2.1 and Definition 2.4, we have

$$f(a) \geq f(G(a)),$$

$$[A(a^\alpha) - G(a^\alpha)] - K_\alpha [A(a) - G(a)] + C \geq \left[A\left(\bar{G}^\alpha\right) - G\left(\bar{G}^\alpha\right)\right] - K_\alpha\left[A\left(\bar{G}\right) - G\left(\bar{G}\right)\right] + C,$$

$$[A(a^\alpha) - G(a^\alpha)] \geq K_\alpha [A(a) - G(a)].$$

Let C_1 enough big to $g(a) = L_\alpha [A(a) - G(a)] - [A(a^\alpha) - G(a^\alpha)] + C_1 \geq 0$ for any $a \in E$. Analogously we can obtain that g is a S-geometrically convex function, and

$$A(a^\alpha) - G(a^\alpha) \leq L_\alpha [A(a) - G(a)].$$

Thus the proof of Inequality (1.1) is completed.

If all λ_i are rational numbers , let $\lambda_i = \frac{k_i}{p}\,(k_i, p \in N)$, $p = k_1 + k_2 + \cdots + k_n$,

$$b = \left(\underbrace{a_1, \ \cdots, \ a_1}_{k_1}, \cdots, \underbrace{a_n, \ \cdots, \ a_n}_{k_n}\right),$$

then from Inequality (1.1), we have

$$K_\alpha\left[\frac{1}{p}\sum\nolimits_{i=1}^{n}k_ia_i-\sqrt[p]{\prod\nolimits_{i=1}^{n}a_i^{k_i}}\right]\le\frac{1}{p}\sum\nolimits_{i=1}^{n}k_ia_i^{\alpha}-\sqrt[p]{\prod\nolimits_{i=1}^{n}a_i^{k_i\alpha}}$$

$$\le L_\alpha\left[\frac{1}{p}\sum\nolimits_{i=1}^{n}k_ia_i-\sqrt[p]{\prod\nolimits_{i=1}^{n}a_i^{k_i}}\right],$$

$$K_\alpha\left[A\left(a,\lambda\right)-G\left(a,\lambda\right)\right]\le A\left(a^\alpha,\lambda\right)-G\left(a^\alpha,\lambda\right)\le L_\alpha\left[A\left(a,\lambda\right)-G\left(a,\lambda\right)\right].$$

If not all λ_i are rational numbers, there exist $\left\{\mu_{i,j}|\mu_{ij}>0,\mu_{ij}\in Q\right\}$ $(i=1,2\cdots,n)\,(j=1,2,\cdots)$, and $\sum_{i=1}^{n}\mu_{ij}=1\,(j=1,2,\cdots)$, $\mu_{i,j}\to\lambda_i\,(j\to+\infty)$. Let $\mu_j=\left(\mu_{1j},\mu_{2j},\cdots\mu_{nj}\right)$, From above proof, we have

$$K_\alpha\left[A\left(a,\mu_j\right)-G\left(a,\mu_j\right)\right]\le A\left(a^\alpha,\mu_j\right)-G\left(a^\alpha,\mu_j\right)\le L_\alpha\left[A\left(a,\mu_j\right)-G\left(a,\mu_j\right)\right].$$

Then $j\to\infty$, Inequality (1.2) holds.

The proof of Inequality (1.2) is completed.

The proof of Theorem 1.2 is omitted here. Reader may complete it by the definition of integral.

The proof of Theorem 1.3 Let $h(a)=\prod\limits_{1\le i_1<\cdots<i_k\le n}(a_{i_1}+a_{i_2}+\cdots+a_{i_k})$, $f\left(a\right)=h(a)\cdot[A(a)]^{-\lambda\binom{n}{k}}$, $a\in R_+^n$, then

$$\frac{\partial f}{\partial a_1}=\frac{h(a)\cdot\sum\limits_{2\le i_2<\cdots<i_k\le n}\frac{1}{a_1+a_{i_2}+\cdots+a_{i_k}}\cdot[A\left(a\right)]^{\lambda\binom{n}{k}}-\frac{\lambda}{n}\binom{n}{k}h\left(a\right)\cdot[A\left(a\right)]^{\lambda\binom{n}{k}-1}}{[A\left(a\right)]^{2\lambda\binom{n}{k}}}$$

$$=h\left(a\right)\cdot\frac{\sum\limits_{2\le i_2<\cdots<i_k\le n}\frac{1}{a_1+a_{i_2}+\cdots+a_{i_k}}\cdot A\left(a\right)-\frac{\lambda}{n}\binom{n}{k}}{[A\left(a\right)]^{\lambda\binom{n}{k}+1}}.$$

Similarly,

$$\frac{\partial f}{\partial a_2}=h\left(a\right)\cdot\frac{\sum\limits_{\substack{1\le i_2<\cdots i_k\le n\\ i_j\ne2,j=2,\cdots,k}}\frac{1}{a_2+a_{i_2}+\cdots+a_{i_k}}\cdot A\left(a\right)-\frac{\lambda}{n}\binom{n}{k}}{[A\left(a\right)]^{\lambda\binom{n}{k}+1}}.$$

So

$$\left(\ln a_1-\ln a_2\right)\left(a_1\frac{\partial f}{\partial a_1}-a_2\frac{\partial f}{\partial a_2}\right)=\left(\ln a_1-\ln a_2\right)\frac{h\left(a\right)}{[A\left(a\right)]^{\lambda\binom{n}{k}+1}}$$

$$\cdot\left[\sum_{2\le i_2<\cdots<i_k\le n}\frac{a_1}{a_1+a_{i_2}+\cdots+a_{i_k}}-\sum_{\substack{1\le i_2<\cdots<i_k\le n\\ i_j\ne2,j=2,\cdots,k}}\frac{a_2}{a_2+a_{i_2}+\cdots+a_{i_k}}\right]A\left(a\right)$$

$$-\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}\frac{\lambda}{n}\binom{n}{k}(a_1-a_2)$$

$$=\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}}}(a_1-a_2)\sum_{3\le i_3<\cdots i_k\le n}\frac{1}{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}$$

$$+\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}}}(a_1-a_2)\sum_{3\le i_2<\cdots i_k\le n}\frac{a_{i_2}+\cdots+a_{i_k}}{(a_1+a_{i2}+\cdots+a_{ik})(a_2+a_{i_2}+\cdots+a_{ik})}$$

$$-\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}\frac{\lambda}{n}\binom{n}{k}(a_1-a_2)$$

$$\ge\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}}}(a_1-a_2)\sum_{3\le i_3<\cdots i_k\le n}\frac{1}{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}$$

$$-\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}\frac{\lambda}{n}\binom{n}{k}(a_1-a_2)$$

$$=(a_1-a_2)\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}\left[\sum_{3\le i_3<\cdots<i_k\le n}\frac{A(a)}{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}-\frac{\lambda}{n}\binom{n}{k}\right]$$

$$=\frac{1}{n}(a_1-a_2)\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}$$

$$\cdot\left[\sum_{3\le i_3<\cdots<i_k\le n}\frac{a_1+a_2+a_3+\cdots+a_n}{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}-\frac{k(k-1)}{n(n-1)}\cdot\frac{n!}{k!(n-k)!}\right]$$

$$=\frac{1}{n}(a_1-a_2)\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}\left[\sum_{3\le i_3<\cdots<i_k\le n}\frac{a_1+a_2+a_3+\cdots+a_n}{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}-\binom{n-2}{k-2}\right]$$

$$\ge\frac{1}{n}(a_1-a_2)\left(\ln a_1-\ln a_2\right)\frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}+1}}\left[\sum_{3\le i_3<\cdots<i_k\le n}\frac{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}{a_1+a_2+a_{i_3}+\cdots+a_{i_k}}-\binom{n-2}{k-2}\right]$$

$$=0.$$

Then f is a S-geometrically convex function with Lemma 2.2. Using the Definition 2.4 and Lemma 2.1, we have

$$f(a)\ge f(\bar{G})$$

$$\Leftrightarrow \frac{h(a)}{[A(a)]^{\lambda\binom{n}{k}}} \geq k^{\binom{n}{k}} \cdot [G(a)]^{(1-\lambda)\binom{n}{k}},$$

$$\Leftrightarrow h(a) \geq k^{\binom{n}{k}} \cdot [A(a)]^{\lambda\binom{n}{k}} \cdot [G(a)]^{(1-\lambda)\binom{n}{k}}.$$

thus the proof of Theorem 1.3 is completed.

4. An Open Problems

Problem 4.1. *What are the best constant* K_α, L_α *which make the inequality* (1.1)(1.4) *holding?*

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