A REFINEMENT OF THE CAUCHY-SCHWARTZ INEQUALITY FOR THE LINEAR AND INNER PRODUCT SPACES

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ABSTRACT. In this paper we refine the Cauchy-Schwartz inequality applicable to both normed linear and inner product spaces. Then we apply the result to some concrete normed spaces.

1. INTRODUCTION

In a normed linear space $(X, \|.\|)$ we define the lower and upper semi-inner products

$$(y, x)_i = \lim_{t \to 0^-} \frac{\|x + ty\|^2 - \|x\|^2}{2t},$$

and

$$(y, x)_s = \lim_{t \to 0^+} \frac{\|x + ty\|^2 - \|x\|^2}{2t},$$

respectively, for every pair $x, y \in X$. We give here a list of some of the properties of these mappings that will be used in this paper, which assuming $p, q \in \{s, i\}$, they are:

- $(x, x)_p = \|x\|^2$ for all $x \in X$.
- $(\alpha x, \beta y)_p = \alpha\beta(x, y)_p$ if $\alpha\beta \geq 0$ and $x, y \in X$.
- $|(x, y)_p| \leq \|x\|^2\|y\|^2$ for all $x, y \in X$.
- $(x + y, z)_p \leq \|x\|\|z\| + (y, z)_p$ for all $x, y, z \in X$.

Then for any two fixed elements $x, y \in X$ we consider the mapping:

$$\gamma_{x,y}(t) := \frac{\|x + 2ty\| - \|x + ty\|}{t} \qquad t \in \mathbb{R}\backslash\{0\}.$$

The main properties of this mapping are described in the following theorem [1].

Theorem 1.1. *Let $(X, \|.\|)$ be a real normed space and $x, y \in X$ two fixed vectors in X. for $p \in \{s, i\}$ we introduce the notation:*

$$\Phi^p_{x,y}(t) := \frac{(y, x + ty)_p}{\|x + ty\|} \qquad \Psi^p_{x,y}(t) := \frac{(x, x + ty)_p}{\|x + ty\|}.$$

Then we have:

(i) *The mapping $\gamma_{x,y}$ is bounded on $\mathbb{R}\backslash\{0\}$ and*

(1.1) $$|\gamma_{x,y}| \leq \|y\| \qquad \forall\ t \in \mathbb{R}\backslash\{0\}.$$

1991 *Mathematics Subject Classification.* 47A63.
Key words and phrases. Inequalities.

(ii) *If x, y are linearly independent, then we have the inequalities*

$$-\|y\| \leq \gamma_{x,y}(u) \leq \Phi^i_{x,y}(u) \qquad \forall\ u < 0, \tag{1.2}$$

and

$$\|y\| \geq \gamma_{x,y}(t) \geq \Phi^s_{x,y}(t) \qquad \forall\ t > 0. \tag{1.3}$$

(iii) *The mapping $\gamma_{x,y}$ is continuous on $\mathbb{R}\backslash\{0\}$ and we have the limits*

$$\lim_{u \to 0^-} \gamma_{x,y}(u) = \frac{(y,x)_i}{\|x\|} \qquad \lim_{t \to 0^+} \gamma_{x,y}(t) = \frac{(y,x)_s}{\|x\|}, \tag{1.4}$$

and

$$\lim_{u \to -\infty} \gamma_{x,y}(u) = -\|y\| \qquad \lim_{u \to +\infty} \gamma_{x,y}(t) = \|y\|. \tag{1.5}$$

if x, y are linearly independent.

iv) *The following inequalities hold*

$$\gamma_{x,y}(t) \leq \Phi^i_{x/2,y}(t) \leq \Phi^s_{x/2,y}(t) \leq \|y\| \qquad \forall\ t > 0, \tag{1.6}$$

and

$$\gamma_{x,y}(u) \geq \Phi^s_{x/2,y}(u) \geq \Phi^i_{x/2,y}(u) \geq -\|y\| \qquad \forall\ u < 0. \tag{1.7}$$

if x, y are linearly independent.

In the next section we use the above theorem to refine the inequality (1.1).

2. Refinement

Proposition 2.1. *Let x, y be linearly independent vectors in a normed linear space $(X, \|.\|)$ and $u < 0 < t$. Then the following inequalities hold:*

$$\begin{aligned} -\|x\|\|y\| \leq \|x\|\Phi^i_{x/2,y}(u) &\leq \|x\|\Phi^s_{x/2,y}(u) \\ &\leq \|x\|\gamma_{x,y}(u) \\ &\leq \|x\|\Phi^i_{x/2,y}(t) \leq \|x\|\Phi^s_{x/2,y}(t) \leq \|x\|\|y\|. \end{aligned}$$

Proof. Follows from Theorem 1.1, in particular from (1.6) and (1.7). □

Remark 2.2. Let (Ω, A, μ) be a measure space, let $p > 1$, and let $L^p(\mu)$ be the Banach space of real valued functions $p-$integrable with respect to μ. It is known [2] that

$$[x, y]_p = \|y\|^{2-p} \int_\Omega |y|^{p-1}\text{sgn}(y)x d\mu,$$

is a semi-inner product on $L^p(\mu)$ generated by the norm $\|x\|_p = (\int_\Omega |x|^p d\mu)^{1/p}$. Since the space $L^p(\mu)$ is smooth, we obtain

$$[x, y]_p = (x, y)_i = (x, y)_s \qquad (x, y \in L^p(\mu)),$$

and applying the preceding theorem with $t > 0$ to any linearly independent vectors $x, y \in L^p(\Omega)$, we get

$$\begin{aligned}
\frac{(\int_\Omega |x+2ty|^p d\mu)^{1/p} - (\int_\Omega |x+ty|^p d\mu)^{1/p}}{t} &\leq \frac{\|\frac{x}{2}+ty\|_p^{2-p}\int_\Omega |\frac{x}{2}+ty|^{p-1}\mathrm{sgn}(y)x d\mu}{(\int_\Omega |\frac{x}{2}+ty|^p d\mu)^{1/p}} \\
&= \frac{\int_\Omega |\frac{x}{2}+ty|^{p-1}\mathrm{sgn}(y)x d\mu}{(\int_\Omega |\frac{x}{2}+ty|^p d\mu)^{\frac{p-1}{p}}} \\
&\leq \left(\int_\Omega |y|^p d\mu\right)^{\frac{1}{p}} = \|y\|_p.
\end{aligned}$$

Also, for $u < 0$ we have

$$\begin{aligned}
\frac{(\int_\Omega |x+2uy|^p d\mu)^{1/p} - (\int_\Omega |x+uy|^p d\mu)^{1/p}}{u} &\geq \frac{\|\frac{x}{2}+uy\|_p^{2-p}\int_\Omega |\frac{x}{2}+uy|^{p-1}\mathrm{sgn}(y)x d\mu}{(\int_\Omega |\frac{x}{2}+uy|^p d\mu)^{1/p}} \\
&= \frac{\int_\Omega |\frac{x}{2}+uy|^{p-1}\mathrm{sgn}(y)x d\mu}{(\int_\Omega |\frac{x}{2}+uy|^p d\mu)^{\frac{p-1}{p}}} \\
&\geq -\left(\int_\Omega |y|^p d\mu\right)^{\frac{1}{p}} = -\|y\|_p.
\end{aligned}$$

Remark 2.3. We consider the space $\ell^1(\mathbb{R})$ consisting of all vectors of the form $x = (x_j)_{j\in\mathbb{N}}$, where $\|x\| = \Sigma_{j=1}^\infty |x_j| < \infty$. It is known from [3] that

$$(x, y)_i = \sum_{k=1}^\infty |y_k| \left(\sum_{y_j\neq 0} \mathrm{sgn}(y_j)x_j - \sum_{y_j=0} |x_j|\right),$$

and

$$(x, y)_s = \sum_{k=1}^\infty |y_k| \left(\sum_{y_j\neq 0} \mathrm{sgn}(y_j)x_j + \sum_{y_j=0} |x_j|\right).$$

Applying the preceding theorem with $t > 0$ to any $x, y \in \ell^1(\mathbb{R})$, we get

$$\begin{aligned}
\gamma_{x,y}(t) &= \frac{\sum_{j=1}^\infty |x_j+2ty_j| - \sum_{j=1}^\infty |x_j+ty_j|}{t} \\
&\leq \Phi_{x,y}^i(t) = \frac{\sum_{k=1}^\infty |\frac{x_k}{2}+ty_k| \left(\sum_{\frac{x_j}{2}+ty_j\neq 0} \mathrm{sgn}(\frac{x_j}{2}+ty_j)y_j - \sum_{\frac{x_j}{2}+ty_j=0} |y_j|\right)}{\sum_{j=1}^\infty |\frac{x_j}{2}+ty_j|} \\
&\leq \Phi_{x,y}^s(t) = \frac{\sum_{k=1}^\infty |\frac{x_k}{2}+ty_k| \left(\sum_{\frac{x_j}{2}+ty_j\neq 0} \mathrm{sgn}(\frac{x_j}{2}+ty_j)y_j + \sum_{\frac{x_j}{2}+ty_j=0} |y_j|\right)}{\sum_{j=1}^\infty |\frac{x_j}{2}+ty_j|} \\
&\leq \|y\| = \sum_{j=1}^\infty |y_j|.
\end{aligned}$$

For $u<0$ we have

$$\begin{aligned}
\gamma_{x,y}(u) &= \frac{\sum_{j=1}^{\infty}|x_j+2uy_j| - \sum_{j=1}^{\infty}|x_j+uy_j|}{t} \\
&\geq \Phi^s_{x,y}(u) = \frac{\sum_{k=1}^{\infty}|\frac{x_k}{2}+uy_k|\left(\sum_{\frac{x_j}{2}+uy_j\neq 0}\operatorname{sgn}(\frac{x_j}{2}+uy_j)y_j + \sum_{\frac{x_j}{2}+uy_j=0}|y_j|\right)}{\sum_{j=1}^{\infty}|\frac{x_j}{2}+uy_j|} \\
&\geq \Phi^i_{x,y}(u) = \frac{\sum_{k=1}^{\infty}|\frac{x_k}{2}+uy_k|\left(\sum_{\frac{x_j}{2}+uy_j\neq 0}\operatorname{sgn}(\frac{x_j}{2}+uy_j)y_j - \sum_{\frac{x_j}{2}+uy_j=0}|y_j|\right)}{\sum_{j=1}^{\infty}|\frac{x_j}{2}+uy_j|} \\
&\geq -\|y\| = -\sum_{j=1}^{\infty}|y_j|.
\end{aligned}$$

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