ABOUT A NEW MEAN

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Abstract. In this paper we introduce a new mean which give new refinements for AM-GM-HM inequalities, and we presents some interesting applications.

1. Introduction

If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$A(a_1, a_2, ..., a_n) = \frac{1}{n}\sum_{k=1}^{n} a_k$$

denotes the arithmetical mean,

$$G(a_1, a_2, ..., a_n) = \sqrt[n]{\prod_{k=1}^{n} a_k}$$

denotes the geometrical mean,

$$H(a_1, a_2, ..., a_n) = \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

denotes the harmonical mean,

$$B(a_1, a_2, ..., a_n) = \ln\left(1 + \sqrt[n]{\prod_{k=1}^{n} (e^{a_k} - 1)}\right)$$

denotes the Bencze's mean (introduced by M. Bencze in 1982, see [1]) and

$$M(a_1, a_2, ..., a_n) = \frac{1}{B\left(\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}\right)}$$

(see [1]).

2. Main Results

Theorem 1. If $a_k > 0\,(k = 1, 2, ..., n)$, then the following inequalities hold:

$$\begin{aligned} H(a_1, a_2, ..., a_n) &\le M(a_1, a_2, ..., a_n) \\ &\le G(a_1, a_2, ..., a_n) \\ &\le B(a_1, a_2, ..., a_n) \le A(a_1, a_2, ..., a_n). \end{aligned}$$

Proof. If $x_k > 0\,(k = 1, 2, ..., n)$, then from AM-GM inequality we have

$$\prod_{k=1}^{n}(1+x_k) = 1 + \sum x_1 + \sum x_1x_2 + ... + \sum x_1x_2...x_{n-1} + \prod_{k=1}^{n} x_k$$
$$\geq 1 + \binom{n}{1}\sqrt[n]{\prod\nolimits_{k=1}^{n} x_k} + \binom{n}{2}\left(\sqrt[n]{\prod\nolimits_{k=1}^{n} x_k}\right)^2 + \cdots$$
$$+ \binom{n}{n-1}\left(\sqrt[n]{\prod\nolimits_{k=1}^{n} x_k}\right)^{n-1} + \left(\sqrt[n]{\prod\nolimits_{k=1}^{n} x_k}\right)^n$$
$$= \left(1 + \sqrt[n]{\prod\nolimits_{k=1}^{n} x_k}\right)^n \quad \text{(the inequality of Huygens)}$$

or

$$\frac{1}{n}\sum_{k=1}^{n}\ln(1+x_k) \geq \ln\left(1 + \sqrt[n]{\prod\nolimits_{k=1}^{n} x_k}\right).$$

If $1 + x_k = e^{a_k}$ $(k = 1, 2, ..., n)$, then we obtain the inequality

$$A(a_1, a_2, ..., a_n) \geq B(a_1, a_2, ..., a_n).$$

We take the following function $f : R \to R$, where $f(x) = \ln(\ln(1 + e^x))$, because

$$f''(x) = \frac{e^x(\ln(1+e^x) - e^x)}{(1+e^x)^2(\ln(1+e^x))^2} < 0,$$

for all $x \in R$, therefore f is concave.

From Jensen's inequality we have

$$\frac{1}{n}\sum_{k=1}^{n} f(\ln x_k) \leq f\left(\frac{1}{n}\sum_{k=1}^{n}\ln x_k\right)$$

for all $x_k > 0$ $(k = 1, 2, ..., n)$, or

$$\sqrt[n]{\prod_{k=1}^{n}\ln(1+x_k)} \leq \ln\left(1 + \sqrt[n]{\prod_{k=1}^{n} x_k}\right).$$

If $1 + x_k = e^{a_k}$ $(k = 1, 2, ..., n)$, then

$$G(a_1, a_2, ..., a_n) \leq B(a_1, a_2, ..., a_n).$$

If in

$$G(a_1, a_2, ..., a_n) \leq B(a_1, a_2, ..., a_n) \leq A(a_1, a_2, ..., a_n)$$

we take $a_k \to \frac{1}{a_k}$ $(k = 1, 2, ..., n)$, then

$$H(a_1, a_2, ..., a_n) \leq M(a_1, a_2, ..., a_n) \leq G(a_1, a_2, ..., a_n).$$

Application 1. For all triangles ABC the following inequalities hold:

(1) $$\frac{12sRr}{s^2 + r^2 + 4Rr} \leq M(a, b, c) \leq \sqrt[3]{4sRr} \leq B(a, b, c) \leq \frac{2s}{3}$$

$$(2)\qquad \frac{3sr}{4R+r} \le M(s-a,s-b,s-c) \le \sqrt[3]{sr^2} \le B(s-a,s-b,s-c) \le \frac{s}{3}$$

$$(3)\qquad 3r \le M(h_a,h_b,h_c) \le \sqrt[3]{\frac{2s^2r^2}{R}} \le B(h_a,h_b,h_c) \le \frac{s^2+r^2+4Rr}{6R}$$

$$(4)\qquad 3r \le M(r_a,r_b,r_c) \le \sqrt[3]{s^2r} \le B(r_a,r_b,r_c) \le \frac{4R+r}{3}$$

$$(5)\qquad \frac{3s}{4R+r} \le M\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \le \sqrt[3]{\frac{s}{r}} \le B\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \le \frac{s}{3r}$$

$$(6)\qquad \frac{3r}{s} \le M\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \le \sqrt[3]{\frac{r}{s}} \le B\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \le \frac{4R+r}{3s}$$

$$(7)\qquad \frac{3r^2}{s^2+r^2-8Rr} \le M\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \le \sqrt[3]{\frac{r^2}{16R^2}} \le B\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \le \frac{2R-r}{6R}$$

$$(8)\qquad \frac{3s^2}{s^2+(4R+r)^2} \le M\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \le \sqrt[3]{\frac{s^2}{16R^2}} \le B\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \le \frac{4R+r}{6R},$$

where A,B,C denote the angles; a,b,c the sides; h_a,h_b,h_c the altitudes; r_a,r_b,r_c the radii of exinscribed circles; r the radius of incircle; R the radius of circumcircle; s the semiperimeter.

These are new refinements for many inequalities published in [2].

Application 2. If S denotes the area of rectangle triangle ABC $(a > b \ge c)$, then (see [2])

$$(9)\qquad \frac{12a^2S^2}{4S^2+a^4} \le M\left(a^2,b^2,c^2\right) \le \sqrt[3]{4a^2S^2} \le B\left(a^2,b^2,c^2\right) \le \frac{2a^2}{3}.$$

Application 3. If V denotes the volume of rectangle paralelipipedon $ABCDA'B'C'D'$ with sides a,b,c and diagonal d then:

$$(10)\qquad \frac{3V^2}{a^2b^2+b^2c^2+c^2a^2} \le M\left(a^2,b^2,c^2\right) \le \sqrt[3]{V^2} \le B\left(a^2,b^2,c^2\right) \le \frac{d^2}{3}$$

$$\begin{aligned}(11)\qquad &\frac{4d^2V^2}{V^2+d^2\left(a^2b^2+b^2c^2+c^2a^2\right)}\\ &\leq M\left(a^2,b^2,c^2,d^2\right)\\ &\leq \sqrt{dV}\leq B\left(a^2,b^2,c^2,d^2\right)\leq \frac{d^2}{2}\quad \text{(see [2]).}\end{aligned}$$

Application 4. In all tetrahedrons $ABCD$:

$$\begin{aligned}(12)\qquad &\frac{3}{4R}\leq M\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c},\frac{1}{h_d}\right)\\ &\leq \frac{1}{\sqrt[4]{h_ah_bh_ch_d}}\leq B\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c},\frac{1}{h_d}\right)\leq\frac{1}{4r}\end{aligned}$$

$$\begin{aligned}(13)\qquad &\frac{3}{2R}\leq M\left(\frac{1}{r_a},\frac{1}{r_b},\frac{1}{r_c},\frac{1}{r_d}\right)\leq\frac{1}{\sqrt[4]{r_ar_br_cr_d}}\\ &\leq B\left(\frac{1}{r_a},\frac{1}{r_b},\frac{1}{r_c},\frac{1}{r_d}\right)\leq\frac{1}{2r}\quad \text{(see [2]).}\end{aligned}$$

(There are new refinements for the Euler's $R\geq 3r$ inequality), where h_a,h_b,h_c,h_d denote the altitudes; r_a,r_b,r_c,r_d the radii of exinscribed spheres; r the radius of insphere; R the radius of circumsphere.

Application 5. The following inequalities hold:

$$(14)\qquad \frac{n}{1+\ln n}\leq M\left(1,2,...,n\right)\leq\sqrt[n]{n!}\leq B\left(1,2,...,n\right)\leq\frac{n+1}{2}$$

$$\begin{aligned}(15)\qquad &\frac{n^2}{2n-1}\leq M\left(1^2,2^2,...,n^2\right)\leq\left(\sqrt[n]{n!}\right)^2\\ &\leq B\left(1^2,2^2,...,n^2\right)\leq\frac{(n+1)(2n+1)}{6}\end{aligned}$$

$$\begin{aligned}(16)\qquad &\frac{8(n-1)n^2}{11n^2-11n-4}\leq M\left(1^3,2^3,...,n^3\right)\leq\left(\sqrt[n]{n!}\right)^3\\ &\leq B\left(1^3,2^3,...,n^3\right)\leq\frac{n(n+1)^2}{4}\quad \text{(see [2]).}\end{aligned}$$

Application 6. If F_k and L_k denote the $k-$th Fibonacci, respective Lucas number, then:

$$\sqrt[n]{\prod_{k=1}^{n} F_k} \leq B\left(F_1, F_2, ..., F_n\right) \leq \frac{F_{n+2}-1}{n} \tag{17}$$

$$\left(\sqrt[n]{\prod_{k=1}^{n} F_k}\right)^2 \leq B\left(F_1^2, F_2^2, ..., F_n^2\right) \leq \frac{F_n F_{n+1}}{n} \tag{18}$$

$$\sqrt[n+1]{\prod_{k=0}^{n} L_{2k+1}} \leq B\left(L_1, L_3, ..., L_{2n+1}\right) \leq \frac{L_{2n+2}-2}{n+1} \tag{19}$$

$$\left(\sqrt[n]{\prod_{k=1}^{n} L_k}\right)^2 \leq B\left(L_1^2, L_2^2, ..., L_n^2\right) \leq \frac{L_n L_{n+1}-2}{n} \quad \text{(see [2]).} \tag{20}$$

Application 7. The following inequalities hold:

$$\sqrt[n+1]{\prod_{k=0}^{n} \binom{n}{k}} \leq B\left(\binom{n}{0}, \binom{n}{1}, ..., \binom{n}{n}\right) \leq \frac{2^n}{n+1} \tag{21}$$

(22)

$$\left(\sqrt[n+1]{\prod_{k=0}^{n} \binom{n}{k}}\right)^2 \leq B\left(\binom{n}{0}^2, \binom{n}{1}^2, ..., \binom{n}{n}^2\right) \leq \frac{\binom{2n}{n}}{n+1} \quad \text{(see [2]).}$$

Application 8. If $x \in (0,1) \cup (1,+\infty)$ then

(23)

$$\frac{(n+1)(x-1)x^n}{x^{n+1}-1} \leq M\left(1, x, x^2, ..., x^n\right) \leq x^{\frac{n}{2}}$$
$$\leq B\left(1, x, x^2, ..., x^n\right) \leq \frac{x^{n+1}-1}{(n+1)(x-1)} \quad \text{(see [2]).}$$

Application 9. The following inequalities hold:

$$\sqrt[n]{n! \prod_{k=1}^{n} k!} \leq B\left(1!1, 2!2, ..., n!n\right) \leq \frac{(n+1)!-1}{n} \tag{24}$$

$$\sqrt[n]{((n+1)!)^2 \prod_{k=1}^{n} k!} \leq B\left(1!2^2, 2!3^2, ..., n!\,(n+1)^2\right) \leq \frac{(n+2)!-2}{n} \quad \text{(see [2]).} \tag{25}$$

Application 10. The following inequalities hold:

(26)

$$\frac{3}{(n+1)(n+2)} \leq M\left(\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, ..., \frac{1}{n(n+1)}\right) \leq \frac{1}{\sqrt[n]{(n+1)(n!)^2}}$$
$$\leq B\left(\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, ..., \frac{1}{n(n+1)}\right) \leq \frac{1}{n+1}$$

(27)
$$\frac{4}{(n+1)(n+2)(n+3)} \leq M\left(\frac{1}{1\cdot 2\cdot 3}, \frac{1}{2\cdot 3\cdot 4}, ..., \frac{1}{n(n+1)(n+2)}\right)$$
$$\leq \sqrt[n]{\frac{2}{(n+2)(n+1)^2(n!)^3}}$$
$$\leq B\left(\frac{1}{1\cdot 2\cdot 3}, \frac{1}{2\cdot 3\cdot 4}, ..., \frac{1}{n(n+1)(n+2)}\right)$$
$$\leq \frac{n+3}{4(n+1)(n+2)}$$

$$\frac{5}{(n+1)(n+2)(n+3)(n+4)}$$
$$\leq M\left(\frac{1}{1\cdot 2\cdot 3\cdot 4}, \frac{1}{2\cdot 3\cdot 4\cdot 5}, ..., \frac{1}{n(n+1)(n+2)(n+3)}\right)$$
$$\leq \sqrt[n]{\frac{12}{(n+3)(n+2)^2(n+1)^3(n!)^4}}$$
$$\leq B\left(\frac{1}{1\cdot 2\cdot 3\cdot 4}, \frac{1}{2\cdot 3\cdot 4\cdot 5}, ..., \frac{1}{n(n+1)(n+2)(n+3)}\right)$$
$$\leq \frac{n^2+6n+11}{3(n+1)(n+2)(n+3)} \quad \text{(see [2])}.$$

Application 11. For all $n \geq 2$ the following inequalities hold:

(28) $$\frac{1}{2}n^{\frac{1}{n-1}} \leq B\left(\sin\frac{\pi}{n}, \sin\frac{2\pi}{n}, ..., \sin\frac{(n-1)\pi}{n}\right) \leq ctg\frac{\pi}{2n}$$

(29) $$\frac{1}{2}n^{\frac{1}{2(n-1)}} \leq B\left(\sin\frac{\pi}{2n}, \sin\frac{2\pi}{2n}, ..., \sin\frac{(n-1)\pi}{2n}\right)$$
$$\leq \frac{1}{2}\left(ctg\frac{\pi}{4n} - 1\right) \quad \text{(see [2])}.$$

Application 12. If $a, b > 0$ then

$$\sqrt{\ln\left(1+\frac{a}{b}\right)\ln\left(1+\frac{b}{a}\right)} \leq \ln 2 \leq \frac{\ln\left(1+\frac{a}{b}\right)+\ln\left(1+\frac{b}{a}\right)}{2} \qquad \text{(see [3])}.$$

Proof. Using Theorem 1 we take $n = 2$, $a_1 = \frac{a}{b}$, $a_2 = \frac{b}{a}$ in

$$G(a_1, a_2) \leq B(a_1, a_2) \leq A(a_1, a_2).$$

3. Generalization

Theorem 2. If $P(x) = \sum_{i=0}^{m} b_i x^{m-i}$ where $b_i \geq 0$ $(i = 0, 1, ..., m)$ and $x_k \geq 0$ $(k = 1, 2, ..., n)$ then the following inequalities hold:

$$\prod_{k=1}^{n} P(x_k) \geq P^n\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right)$$

(A generalization of Huygens inequality) (see [4]).

Proof. In [4] is presented a proof. Now we present a new proof, using mathematical induction.

For $n = 2$, we have:

$$\left(\sum_{i=0}^{m} b_i x_1^{m-i}\right)\left(\sum_{i=0}^{m} b_i x_2^{m-i}\right) \geq \left(\sum_{i=0}^{m} b_i \left(\sqrt{x_1 x_2}\right)^{m-i}\right)^2,$$

but this holds from

$$b_i b_j x_1^{m-i} x_2^{m-j} + b_i b_j x_2^{m-i} x_1^{m-j} \geq 2 b_i b_j \left(\sqrt{x_1 x_2}\right)^{m-i} \left(\sqrt{x_1 x_2}\right)^{m-j},$$

for all $0 \leq i < j \leq m$.

We suppose it is true for n and we prove for $2n$, so

$$\begin{aligned}\prod_{k=1}^{2n} P(x_k) &= \left(\prod_{k=1}^{n} P(x_k)\right)\left(\prod_{k=n+1}^{2n} P(x_k)\right) \\ &\geq P^n\left(\sqrt[n]{\prod\nolimits_{k=1}^{n} x_k}\right) P^n\left(\sqrt[n]{\prod\nolimits_{k=n+1}^{2n} x_k}\right) \\ &\geq P^{2n}\left(\sqrt[2n]{\prod\nolimits_{k=1}^{2n} x_k}\right)\end{aligned}$$

is true.

If

$$x_{n+2} = x_{n+3} = \ldots = x_{2n} = \sqrt[n+1]{\prod\nolimits_{k=1}^{n+1} x_k},$$

then

$$\prod_{k=1}^{n+1} P(x_k)\, P^{n-1}\left(\sqrt[n+1]{\prod\nolimits_{k=1}^{n+1} x_k}\right) \geq P^{2n}\left(\sqrt[2n]{\prod\nolimits_{k=1}^{n+1} x_k \left(\prod\nolimits_{k=1}^{n+1} x_k\right)^{\frac{n-1}{n+1}}}\right)$$

or

$$\begin{aligned}\prod_{k=1}^{n+1} P(x_k)\, P^{n-1}\left(\sqrt[n+1]{\prod\nolimits_{k=1}^{n+1} x_k}\right) &\geq P^{2n}\left(\sqrt[2n]{\left(\prod\nolimits_{k=1}^{n+1} x_k\right)^{\frac{2n}{n+1}}}\right) \\ &= P^{2n}\left(\sqrt[n+1]{\prod\nolimits_{k=1}^{n+1} x_k}\right)\end{aligned}$$

or

$$\prod_{k=1}^{n+1} P(x_k) \geq P^{n+1}\left(\sqrt[n+1]{\prod\nolimits_{k=1}^{n+1} x_k}\right)$$

so it is true for $n + 1$.

Remark 1. If $P(x) = 1 + x$ then the Huygens inequality holds.

Remark 2. If $x_k > 0$ $(k = 1, 2, ..., n)$ then the following inequality holds:

$$\sum_{k=1}^{n} P(x_k) \geq nP\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right)$$

(see [4]).

Theorem 3. If $a_k > 0$ $(k = 1, 2, ..., n)$ and the function $\ln(\ln P(e^x))$ is concave where $P(x) = \sum_{i=0}^{m} b_i x^{m-i}$ and $b_i \geq 0$ $(i = 0, 1, ..., m)$ is a bijective polynomial then the following inequalities hold:

$$\begin{aligned} H(a_1, a_2, ..., a_n) &\leq M_P(a_1, a_2, ..., a_n) \leq G(a_1, a_2, ..., a_n) \\ &\leq B_P(a_1, a_2, ..., a_n) \leq A(a_1, a_2, ..., a_n), \end{aligned}$$

where

$$B_P(a_1, a_2, ..., a_n) = \ln P\left(\sqrt[n]{\prod_{k=1}^{n} P^{-1}(e^{a_k})}\right)$$

is the generalized Bencze $P-$ mean and

$$M_P(a_1, a_2, ..., a_n) = \frac{1}{B_p\left(\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}\right)}.$$

Proof. Using Theorem 2,

$$\prod_{k=1}^{n} P(x_k) \geq P^n\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right)$$

or

$$\frac{1}{n}\sum_{k=1}^{n} \ln P(x_k) \geq \ln P\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right).$$

If $P(x_k) = e^{a_k}$ or $x_k = P^{-1}(e^{a_k})$ $(k = 1, 2, ..., n)$, then

$$A(a_1, a_2, ..., a_n) \geq B_P(a_1, a_2, ..., a_n).$$

Because the function $f : R \to R$ where $f(x) = \ln(\ln P(e^x))$ is concave then for all $y_k \in R$ $(k = 1, 2, ..., n)$ the Jensen's inequality holds

$$\frac{1}{n}\sum_{k=1}^{n} f(y_k) \leq f\left(\sum_{k=1}^{n} y_k\right)$$

or

$$\sqrt[n]{\prod_{k=1}^{n} \ln P(e^{y_k})} \leq \ln P\left(e^{\frac{1}{n}\sum_{k=1}^{n} y_k}\right).$$

If $y_k = \ln x_k$ where $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sqrt[n]{\prod_{k=1}^{n} \ln P(x_k)} \leq \ln P\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right).$$

In this case we take $P(x_k) = e^{a_k}$ or $x_k = P^{-1}(e^{a_k})$ $(k = 1, 2, ..., n)$, then

$$G(a_1, a_2, ..., a_n) \leq B_P(a_1, a_2, ..., a_n).$$

If in inequalities

$$G(a_1, a_2, ..., a_n) \leq B_P(a_1, a_2, ..., a_n) \leq A(a_1, a_2, ..., a_n)$$

we take $a_k \to \frac{1}{a_k}$ $(k = 1, 2, ..., n)$, then

$$H(a_1, a_2, ..., a_n) \leq M_P(a_1, a_2, ..., a_n) \leq G(a_1, a_2, ..., a_n).$$

Remark 3. If $P(x) = 1 + x$ then we obtain Theorem 1.

Remark 4. If $x_k > 0\,(k = 1, 2, ..., n)$ and the function $\ln(\ln P(e^x))$ is concave where $P(x) = \sum_{i=0}^{m} b_i x^{m-i}$ and $b_i \geq 0$ $(i = 0, 1, ..., m)$, then

$$\sqrt[n]{\prod_{k=1}^{n} \ln P(x_k)} \leq \ln P\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right) \leq \frac{1}{n}\sum_{k=1}^{n} \ln P(x_k).$$

Remark 5. If $\alpha > 0$ and $x_k > 0\,(k = 1, 2, ..., n)$, then the following inequalities hold:

$$\sqrt[n]{\prod_{k=1}^{n} \ln(1 + x_k^{\alpha})} \leq \ln\left(1 + \sqrt[n]{\prod_{k=1}^{n} x_k^{\alpha}}\right) \leq \frac{1}{n}\sum_{k=1}^{n} \ln(1 + x_k^{\alpha}).$$

Proof. The function $\ln(\ln(1 + e^{\alpha x}))$ is concave, so we take $P(x) = 1 + x^{\alpha}$ in Remark 4.

4. The Pondered Extension

Theorem 4. If $P(x) = \sum_{i=0}^{m} b_i x^{m-i}$ where $b_i \geq 0$ $(i = 0, 1, ..., m)$ and $x_k, \alpha_k > 0$ $(k = 1, 2, ..., n)$ then

$$\prod_{k=1}^{n} P^{\alpha_k}(x_k) \geq \left(P\left(\left(\prod_{k=1}^{n} x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}}\right)\right)^{\sum_{k=1}^{n}\alpha_k}$$

Proof. See Theorem 5 in [5].

Remark 6. If $P(x) = \sum_{i=0}^{m} b_i x^{m-i}$ where $b_i \geq 0$ $(i = 0, 1, ..., m)$ and $x_k, \alpha_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{k=1}^{n} \alpha_k P(x_k) \geq \left(\sum_{k=1}^{n}\alpha_k\right) P\left(\left(\prod_{k=1}^{n} x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}}\right).$$

Proof. See Theorem 2 in [5].

Theorem 5. If the function $\ln(\ln P(e^x))$ is concave where $P(x) = \sum_{i=0}^{m} b_i x^{m-i}$ and $b_i \geq 0$ $(i = 0, 1, ..., m)$ and if $x_k, \alpha_k > 0$ $(k = 1, 2, ..., n)$

then

$$\left(\prod_{k=1}^{n}\left(\ln P\left(x_k\right)\right)^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}} \le \ln P\left(\left(\prod_{k=1}^{n}x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}}\right) \le \frac{\sum_{k=1}^{n}\alpha_k \ln P\left(x_k\right)}{\sum_{k=1}^{n}\alpha_k}.$$

Proof. From Theorem 4,

$$\frac{\sum_{k=1}^{n}\alpha_k \ln P\left(x_k\right)}{\sum_{k=1}^{n}\alpha_k} \ge \ln P\left(\left(\prod_{k=1}^{n}x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}}\right).$$

The function $\ln\left(\ln P\left(e^x\right)\right)$ is concave, therefore from Jensen's inequality

$$\left(\prod_{k=1}^{n}\left(\ln P\left(x_k\right)\right)^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}} \le \ln P\left(\left(\prod_{k=1}^{n}x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n}\alpha_k}}\right).$$

Remark 7. If $x_k, a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\prod_{k=1}^{n}\left(\ln\left(1+x_k^{\frac{1}{a_k}}\right)\right)^{a_k} \ge \left(\ln\left(1+\left(\prod_{k=1}^{n}x_k\right)^{\frac{1}{\sum_{k=1}^{n}a_k}}\right)\right)^{\sum_{k=1}^{n}a_k}.$$

Proof. By letting $x_k = y_k^{\sum_{k=1}^{n}a_k}$ $(k = 1, 2, ..., n)$ and $\lambda_k = \frac{a_k}{\sum_{k=1}^{n}a_k}$ $(k = 1, 2, ..., n)$ we get the inequality in the more convenient form

$$\prod_{k=1}^{n}\left(\ln\left(1+y_k^{\frac{1}{\lambda_k}}\right)\right)^{\lambda_k} \ge \ln\left(1+\prod_{k=1}^{n}y_k\right).$$

In order to prove this inequality, we pick a positive number Y and check the validity of this inequality for all positive real numbers y_k $(k = 1, 2, ..., n)$ satisfying $\prod_{k=1}^{n}y_k = Y$.

The inequality holds true for $n = 1$. Thus we let further on $n \ge 2$. $K(Y)$ denote the set $\left\{(y_1, y_2, ..., y_n) \in \overline{R}_{\ge 0}^{n} \text{ for which } \prod_{k=1}^{n}y_k = Y\right\}$.

We now employ the method of Lagrangian multipliers for our purpose.

1). The interior of $K(Y)$.

Letting

$$\Phi\left(y_1, y_2, ..., y_n, \lambda\right) = \sum_{k=1}^{n}\lambda_k \ln\left(\ln\left(1+y_k^{\frac{1}{\lambda_k}}\right)\right) - \lambda\left(\prod_{k=1}^{n}y_k - Y\right),$$

we infer as a necessary condition for interior critical points of Φ: $\frac{\partial\Phi}{\partial y_k} = 0$ $(k = 1, 2, ..., n)$, that is

$$\frac{y_k^{\frac{1}{\lambda_k}-1}}{\left(1 + y_k^{\frac{1}{\lambda_k}}\right) \ln \left(1 + y_k^{\frac{1}{\lambda_k}}\right)} - \lambda \prod_{\substack{j=1 \\ j \neq k}}^{n} y_j = 0 \;\; (k = 1, 2, ..., n)$$

or $f\left(y_k^{\frac{1}{\lambda_k}}\right) = \frac{1}{\lambda Y}$ $(k = 1, 2, ..., n)$ where $f(z) = \frac{(1+z)\ln(1+z)}{z}$, $z > 0$. Therefore, there has to hold $f\left(y_1^{\frac{1}{\lambda_1}}\right) = ... = f\left(y_n^{\frac{1}{\lambda_n}}\right)$.

Because $f'(z) = \frac{z - \ln(1+z)}{z^2} > 0$ as $z > 0$ we get that $f(z)$ increases strictly. This implies $y_1^{\frac{1}{\lambda_1}} = ... = y_n^{\frac{1}{\lambda_n}} = t$ or $y_k = t^{\lambda_k}$ $(k = 1, 2, ..., n)$.

Because of $\prod_{k=1}^{n} y_k = t^{\sum_{k=1}^{n} \lambda_k} = t$ we have $z = Y$ and

$$\prod_{k=1}^{n} (\ln(1+Y))^{\lambda_k} \geq \ln(1+Y),$$

which clearly holds true.

2). The boundary of $K(Y)$. There is $y_i = 0$ for at least one i, whence $y_j = \infty$ for at least one j.

But then the claimed inequality is evident and the proof is complete.

Theorem 6. If in condition of Theorem 5 the polynomial P is bijective and $a_k, \alpha_k > 0$ $(k = 1, 2, ..., n)$, thenthe following inequalities hold:

$$\begin{aligned} \overline{H}(a_1, a_2, ..., a_n) &\leq \overline{M_P}(a_1, a_2, ..., a_n) \leq \overline{G}(a_1, a_2, ..., a_n) \\ &\leq \overline{B_P}(a_1, a_2, ..., a_n) \leq \overline{A}(a_1, a_2, ..., a_n) \end{aligned}$$

where

$$\begin{aligned} \overline{H}(a_1, a_2, ..., a_n) &= \frac{\sum_{k=1}^{n} \alpha_k}{\sum_{k=1}^{n} \frac{\alpha_k}{a_k}}, \\ \overline{G}(a_1, a_2, ..., a_n) &= \left(\prod_{k=1}^{n} a_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^{n} \alpha_k}}, \\ \overline{A}(a_1, a_2, ..., a_n) &= \frac{\sum_{k=1}^{n} \alpha_k a_k}{\sum_{k=1}^{n} \alpha_k} \end{aligned}$$

are the pondered harmonical, geomertrical and arithmetical means,

$$\overline{B_P}(a_1, a_2, ..., a_n) = \ln\left(P\left(\prod_{k=1}^{n} \left(P^{-1}(a_k)\right)^{\alpha_k}\right)\right)^{\frac{1}{\sum_{k=1}^{n} \alpha_k}}$$

is the pondered Bencze's P-mean and

$$\overline{M_P}\left(a_1,a_2,...,a_n\right)=\frac{1}{\overline{B_P}\left(\frac{1}{a_1},\frac{1}{a_2},...,\frac{1}{a_n}\right)}.$$

Proof. In Theorem 5 we take $x_k=P^{-1}\left(e^{a_k}\right)\ (k=1,2,...,n)$.

5. The Extension for Log-Convex and Increasing Functions

Theorem 7. If the function $f:R\to[1,+\infty)$ is log-convex and increasing and the function $\ln f\left(e^x\right)$ is log-concave, then for all $x_k,\alpha_k>0\ (k=1,2,...,n)$ the following inequalities hold:

$$\left(\prod_{k=1}^n\left(\ln f\left(x_k\right)\right)^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^n\alpha_k}}\le\ln f\left(\left(\prod_{k=1}^n x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^n\alpha_k}}\right)$$
$$\le\frac{\sum_{k=1}^n\alpha_k\ln f\left(x_k\right)}{v\alpha_k}.$$

Proof. Because f is log-convex and increasing from Jensen's inequality,

$$\frac{\sum_{k=1}^n\alpha_k\ln f\left(x_k\right)}{\sum_{k=1}^n\alpha_k}\ge\ln f\left(\frac{\sum_{k=1}^n\alpha_kx_k}{\sum_{k=1}^n\alpha_k}\right)\ge\ln f\left(\left(\prod_{k=1}^n x_k^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^n\alpha_k}}\right)$$

for the rest of proof see Theorem 5.

Theorem 8. If the function $f:R\to[1,+\infty)$ is log-convex, increasing and bijective, the function $\ln f\left(e^x\right)$ is log-concave then for all $a_k,\alpha_k>0\ (k=1,2,...,n)$ the following inequalities hold:

$$\overline{H}\left(a_1,a_2,...,a_n\right)\le\overline{M_f}\left(a_1,a_2,...,a_n\right)\le\overline{G}\left(a_1,a_2,...,a_n\right)$$
$$\le\overline{B_f}\left(a_1,a_2,...,a_n\right)\le\overline{A}\left(a_1,a_2,...,a_n\right)$$

where

$$\overline{B_f}\left(a_1,a_2,...,a_n\right)=\ln f\left(\left(\prod_{k=1}^n\left(f^{-1}\left(a_k\right)\right)^{\alpha_k}\right)^{\frac{1}{\sum_{k=1}^n\alpha_k}}\right),$$

denote the pondered Bencze's f-mean and

$$\overline{M_f}\left(a_1,a_2,...,a_n\right)=\frac{1}{\overline{B_f}\left(\frac{1}{a_1},\frac{1}{a_2},...,\frac{1}{a_n}\right)}.$$

Proof. In Theorem 7 we take $x_k=f^{-1}\left(a_k\right)\ (k=1,2,...,n)$, etc.

Remark 8. Theorem 7 and Theorem 8 are true when log-convexity and increasing condition is replaced by geometrically convex (see [7]).

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