DIFFERENTIAL CRITERION OF N-DIMENSIONAL GEOMETRICALLY CONVEX FUNCTIONS

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ABSTRACT. This paper presents a differential criterion of n dimensional geometrically convex functions, and gives some applications.

1. Introduction to Geometrically Convex Functions

Since the theory of convex functions established by Danish mathematician, J. L. W. V. Jensen(1859-1925), last century, the research on convex functions has been lasted for more than one hundred years. However, the research on geometrically convex functions only appear in [1]-[9], especially the recent years' research on it has shown its importance. Just as stated in the preliminary marks of book [3], convex functions and geometrically convex functions are parallel in definitions; both have their advantages when they are used to prove an inequality. Sometimes it is easier to prove an inequality by the property of convex functions, or vice versa. Similarly the result could be better proved by convex functions than by geometrically convex functions or vice versa, while they are used to establish a new inequality. Therefore, geometrically convex functions and convex functions cannot be separated or neglected as they are used as two proofs and tools to discover inequalities.

The definition of geometrically convex functions on one dimension has been formally stated in papers [1] and [2]. n-dimensional geometrically convex functions and S-geometrically convex functions have been defined in papers [4] [5] and [6]. Readers may read paper or book [1]- [9] for more reference.

Throughout the paper we assume R^n be the n-dimensional Euclidean Space,

$$R_+^n = \{(x_1, x_2, \cdots, x_n) \,| x_i > 0, i = 1, 2, \cdots, n\},$$

and

$$e^X = (e^{x_1}, e^{x_2}, \cdots, e^{x_n}), \quad X \cdot Y = (x_1 y_1, x_2 y_2, \cdots, x_n y_n),$$

where $X = (x_1, x_2, \cdots, x_n) \in R^n, Y = (y_1, y_2, \cdots, y_n) \in R^n$. They are defined respectively by

$$X^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha), \quad \ln X = (\ln x_1, \ln x_2, \cdots, \ln x_n),$$

where $X \in R_+^n, \alpha > 0$. If $f : R^n \to R$ if twice differentiable, $f_{ij}''(x)$ means $\frac{\partial^2 f(x)}{\partial x_i \partial x_j}$ with $i, j = 1, 2, \cdots, n$.

Definition 1.1. ([1])([2])([3])([7]) *Let $I \subseteq (0, +\infty)$, $f : I \to (0, +\infty)$ be a continuous function. Then f is called a geometrically convex function on I, if there exists $m \in N, m \geq 2$, such that one of the following four inequalities holds for any $x_1, x_2, \cdots, x_m \in I$, $\alpha, \beta, \lambda_1, \lambda_2, \cdots, \lambda_m > 0$ with $\alpha + \beta = 1$, $\sum_{i=1}^m \lambda_i = 1$,*

$$f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1) f(x_2)}. \tag{1.1}$$

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$$f\left(x_1^{\alpha}x_2^{\beta}\right) \le f^{\alpha}\left(x_1\right) f^{\beta}\left(x_2\right). \tag{1.2}$$

$$f\left(\sqrt[m]{\prod_{i=1}^{m} x_i}\right) \le \sqrt[m]{\prod_{i=1}^{m} f\left(x_i\right)}. \tag{1.3}$$

$$f\left(\prod_{i=1}^{m} x_i^{\lambda_i}\right) \le \prod_{i=1}^{m} f^{\lambda_i}\left(x_i\right). \tag{1.4}$$

Further f is called a geometrically concave function on I if one of four inequalities (1.1)-(1.4) *is inverse.*

[3] prove that (1.1)-(1.4) are equivalent to each other.

Definition 1.2. ([3])([4])([5]) *Let $E \subseteq R_+^n$. Then E is said to be a logarithm convex set if $X^{\alpha}Y^{\beta} \in E$ for any $X, Y \in E$, $\alpha, \beta > 0$ with $\alpha + \beta = 1$.*

Remark 1.1. Let $E \subseteq R_+^n$, $\ln E = \{\ln X | X \in E\}$. Then $X, Y \in E$ if only if $\ln X, \ln Y \in \ln E$, hence $X^{\alpha}Y^{\beta} \in E$ if only if $\alpha \ln X + \beta \ln Y \in \ln E$, E is a logarithm convex set if only if $\ln E$ is a convex set.

Definition 1.3. ([3])([4])([5])([6]) *Let $E \subseteq R_+^n$ be a logarithm convex set, $f : E \to (0, +\infty)$ be a continuous function. Then f is called a geometrically convex function on E, if there exists $m \in N, m \ge 2$, such that one of the following four inequalities holds for any $X_1, X_2, \cdots, X_m \in E$, $\alpha, \beta, \lambda_1, \lambda_2, \cdots, \lambda_m > 0$ with $\alpha + \beta = 1$, $\sum_{i=1}^{m} \lambda_i = 1$,*

$$f\left(X_1^{\frac{1}{2}} \cdot X_2^{\frac{1}{2}}\right) \le \sqrt{f(X_1) f(X_2)}. \tag{1.5}$$

$$f\left(X_1^{\alpha}X_2^{\beta}\right) \le f^{\alpha}\left(X_1\right) f^{\beta}\left(X_2\right). \tag{1.6}$$

$$f\left(\left(\prod_{i=1}^{m} X_i\right)^{\frac{1}{m}}\right) \le \sqrt[m]{\prod_{i=1}^{m} f\left(X_i\right)}. \tag{1.7}$$

$$f\left(\prod_{i=1}^{m} X_i^{\lambda_i}\right) \le \prod_{i=1}^{m} f^{\lambda_i}\left(X_i\right). \tag{1.8}$$

Further f is called a geometrically concave function on E if one of four inequalities (1.5)-(1.8) *is inverse.*

[1] proves that (1.5)-(1.8) are equivalent to each other.

2. Main Results

Let $H \subseteq R^n$, $\phi : H \to R$ be twice differentiable. Then Hessian matrix of ϕ is defined as

$$L(x) = \begin{pmatrix} \phi''_{11} & \phi''_{12} & \cdots & \phi''_{1n} \\ \phi''_{21} & \phi''_{22} & \cdots & \phi''_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi''_{n1} & \phi''_{n2} & \cdots & \phi''_{nn} \end{pmatrix}$$

Lemma 2.1. ([11]) *Let $H \subseteq R^n$ be convex and open. Then ϕ is convex (concave) if only if its Hessian matrix is positive (negative) semi-definite for all $(x_1, x_2, \cdots, x_n) \in H$.*

Lemma 2.2. ([3]) *(1) Let f be geometrically convex (concave) function on $E \subseteq R_+^n$, $\ln E = \{\ln X | X \in E\}$, $g(X) = \ln f\left(e^X\right)$ with $X \in \ln E$. Then g is convex (concave) function.*

(2) Let g be convex (concave) function on $H \subseteq R^n$, $e^H = \left\{e^X | X \in H\right\}$, $f(X) = e^{g(\ln X)}$ with $X \in e^H$. Then f is a geometrically convex (concave) function.

Theorem 2.1. *Let E be a a logarithm convex set, $f : E \subseteq R_+^n \to (0, +\infty)$ be twice differentiable. Then f is geometrically convex (concave) function if only if matrix*

$$\Omega = \begin{pmatrix} f\ f''_{11} + \frac{f}{x_1} f'_1 - (f'_1)^2 & f\ f''_{12} - f'_1\ f'_2 & \cdots & f\ f''_{1n} - f'_1 f'_n \\ f\ f''_{21} - f'_1\ f'_2 & f\ f''_{22} + \frac{f}{x_2} f'_2 - (f'_2)^2 & \cdots & f\ f''_{2n} - f'_2\ f'_n \\ \cdots & \cdots & \cdots & \cdots \\ f\ f''_{n1} - f'_1\ f'_n & f\ f''_{n2} - f'_2\ f'_n & \cdots & f\ f''_{nn} + \frac{f}{x_n} f'_n - (f'_n)^2 \end{pmatrix}$$

is positive (negative) semi-definite.

Proof. Let $g(X) = \ln f\left(e^X\right), X \in \ln E$. Then $\left(\frac{\partial^2\left[\ln f\left(e^Y\right)\right]}{\partial y_i \partial y_j}\right) \in R^{n\times n}$ and

$$\frac{\partial\left(\ln f\left(e^Y\right)\right)}{\partial y_i} = \frac{f'_i\left(e^Y\right)}{f\left(e^Y\right)} e^{y_i}, i = 1, 2, \cdots, n.$$

$$\frac{\partial^2\left(\ln f\left(e^Y\right)\right)}{\partial y_i^2} = \frac{\left(e^{y_i} f'_i\left(e^Y\right) + e^{2y_i} f''_{ii}\left(e^Y\right)\right) f\left(e^Y\right) - e^{y_i} f'_i\left(e^Y\right)\ f'_i\left(e^Y\right) e^{y_i}}{f^2\left(e^Y\right)},$$

$$\frac{\partial^2\left(\ln f\left(e^Y\right)\right)}{\partial y_i^2} = \frac{\left(e^{y_i} f'_i\left(e^Y\right) + e^{2y_i} f''_{ii}\left(e^Y\right)\right) f\left(e^Y\right) - e^{2y_i}\left(f'_i\left(e^Y\right)\right)^2}{f^2\left(e^Y\right)}$$

and

$$\frac{\partial^2\left(\ln f\left(e^Y\right)\right)}{\partial y_i \partial y_j} = \frac{e^{y_i} e^{y_j}\ f''_{ij}\left(e^Y\right)\ f\left(e^Y\right) - e^{y_i} e^{y_j}\ f'_i\left(e^Y\right)\ f'_j\left(e^Y\right)}{f^2\left(e^Y\right)},$$

where $i, j = 1, 2, \cdots, n, i \neq j$. So

$$\left(\frac{\partial^2\left[\ln f\left(e^Y\right)\right]}{\partial y_i \partial y_j}\right) = \left(f\left(e^Y\right)\right)^{-2} \tag{2.1}$$

$$\begin{pmatrix} e^{2y_1}\left[f\ f''_{11} + \frac{f}{e^{y_1}} f'_1 - (f'_1)^2\right] & e^{y_1+y_2}\left[f\ f''_{12} - f'_1\ f'_2\right] & \cdots & e^{y_1+y_n}\left[f\ f''_{1n} - f'_1\ f'_n\right] \\ e^{y_1+y_2}\left[f\ f''_{21} - f'_1\ f'_2\right] & e^{2y_2}\left[f\ f''_{22} + \frac{f}{e^{y_2}} f'_2 - (f'_2)^2\right] & \cdots & e^{y_2+y_n}\left[f\ f''_{2n} - f'_2\ f'_n\right] \\ \cdots & \cdots & \cdots & \cdots \\ e^{y_1+y_n}\left[f\ f''_{n1} - f'_1\ f'_n\right] & e^{y_2+y_n}\left[f\ f''_{n2} - f'_2\ f'_n\right] & \cdots & e^{2y_n}\left[f\ f''_{nn} + \frac{f}{e^{y_n}} f'_n - (f'_n)^2\right] \end{pmatrix}$$

Let $1 \leq k \leq n, k \in N$, $\det A$ denote the determinant of matrix A. Then the kth-order principal submatrixes of $\left(\frac{\partial^2\left[\ln f\left(e^Y\right)\right]}{\partial y_i \partial y_j}\right)$ is

$$\left(f\left(e^Y\right)\right)^{-2k}$$

$$\cdot \det \begin{pmatrix} e^{2y_1}\left[f\ f''_{11} + \frac{f}{e^{y_1}} f'_1 - (f'_1)^2\right] & e^{y_1+y_2}\left[f\ f''_{12} - f'_1\ f'_2\right] & \cdots & e^{y_1+y_n}\left[f\ f''_{1k} - f'_1 f'_k\right] \\ e^{y_1+y_2}\left[f\ f''_{21} - f'_1\ f'_2\right] & e^{2y_2}\left[f\ f''_{22} + \frac{f}{e^{y_2}} f'_2 - (f'_2)^2\right] & \cdots & e^{y_2+y_n}\left[f\ f''_{2k} - f'_2\ f'_k\right] \\ \cdots & \cdots & \cdots & \cdots \\ e^{y_1+y_k}\left[f\ f''_{k1} - f'_1\ f'_k\right] & e^{y_2+y_k}\left[f\ f''_{k2} - f'_2\ f'_k\right] & \cdots & e^{2y_n}\left[f\ f''_{kk} + \frac{f}{e^{y_k}} f'_k - (f'_k)^2\right] \end{pmatrix}$$

$$= \left(f\left(e^Y\right)\right)^{-2k} \cdot e^{2(y_1+y_2+\cdots+y_k)}$$

$$\cdot\det\begin{pmatrix} f\ f''_{11}+\frac{f}{e^{y_1}}f'_1-(f'_1)^2 & f\ f''_{12}-f'_1\ f'_2 & \cdots & f\ f''_{1k}-f'_1 f'_k \\ f\ f''_{21}-f'_1\ f'_2 & f\ f''_{22}+\frac{f}{e^{y_2}}f'_2-(f'_2)^2 & \cdots & f\ f''_{2k}-f'_2\ f'_k \\ \cdots & \cdots & \cdots & \cdots \\ f\ f''_{k1}-f'_1\ f'_k & f\ f''_{k2}-f'_2\ f'_k & \cdots & f\ f''_k+\frac{f}{e^{y_k}}f'_k-(f'_k)^2 \end{pmatrix}$$

Let $e^{y_i}=x_i, i=1,2,\cdots,n$. We know $\left(\frac{\partial^2\left[\ln f\left(e^Y\right)\right]}{\partial y_i\partial y_j}\right)$ is positive (negative) semi-definite if only if Ω is positive (negative) semi-definite. By Lemma 2.1 and Lemma 2.2, Ω is positive (negative) semi-definite, if only if g is a convex (concave) function and f is a geometrically convex (concave) function. ∎

Theorem 2.2. *Let $f: R_+^2\to R_+$ be homogeneous form of degree m and twice continuously differentiable. Then f is a geometrically convex (concave) function if only if $\frac{\partial^2(\ln f(x_1,x_2))}{\partial x_1\partial x_2}\le(\ge)\,0$ or $f'_1\cdot f'_2-f\cdot f''_{12}\ge(\le)\,0$.*

Proof. For all $t\ge 0$, we have $f(tx_1,tx_2)=t^m f(x_1,x_2)$, so

$$x_1 f'_1(tx_1,tx_2)+x_2 f'_2(tx_1,tx_2)=mt^{m-1}f(x_1,x_2)\,. \tag{2.2}$$

Find the derivative on both sides of (2.2) with respect to x_1,

$$f'_1+xf''_{11}+yf''_{12}=m\cdot f'_1,$$

$$f''_{11}=-\frac{y}{x}f''_{12}+\frac{m-1}{x}f'_1.$$

hence

$$\det\begin{pmatrix} f\ f''_{11}+\frac{f\ f'_1}{x_1}-(f'_1)^2 & f\ f''_{12}-f'_1f'_2 \\ f\ f''_{12}-f'_1f'_2 & f\ f''_{22}+\frac{f\ f'_2}{x_2}-(f'_2)^2\end{pmatrix}$$

$$=\det\begin{pmatrix} \frac{x_2}{x_1}(f'_1f'_2-f\ f''_{12}) & f\ f''_{12}-f'_1f'_2 \\ f\ f''_{12}-f'_1f'_2 & \frac{x_1}{x_2}(f'_1f'_2-f\ f''_{12})\end{pmatrix}=0.$$

Further

$$f\ f''_{11}+\frac{f}{x_1}f'_1-\left(f'_1\right)^2=-\frac{x_2}{x_1}f\ f''_{12}+\frac{m}{x_1}f\ f_1-\left(f'_1\right)^2,$$

$$-\frac{x_2}{x_1}f\ f''_{12}+\frac{x_1f'_1+x_2f'_2}{x_1}f'_1-\left(f'_1\right)^2$$

$$=\frac{x_2}{x_1}\left(f'_1f'_2-f\ f''_{12}\right)=-\frac{x_2}{x_1}\left(\ln f\left(x_1,x_2\right)\right)''_{12}.$$

So that $f\ f''_{11}+\frac{f}{x_1}f'_1-(f'_1)^2\ge(\le)\,0$ if only if $(\ln f(x_1,x_2))''_{12}\le(\ge)\,0$, f is a geometrically convex (concave) function if only if $\frac{\partial^2(\ln f(x_1,x_2))}{\partial x_1\partial x_2}\le(\ge)\,0$ according to Theorem 2.1. ∎

3. Some Applications

Example 3.1. *(H ö lder-Inequality) Let $x_i,y_i\in R_+\,(i=1,2,\cdots,n)$, $p>1,q>1,\frac{1}{p}+\frac{1}{q}=1$. Then*

$$(x_1^p+x_2^p+\cdots+x_n^p)^{\frac{1}{p}}(y_1^q+y_2^q+\cdots+y_n^q)^{\frac{1}{q}}\ge x_1y_1+x_2y_2+\cdots+x_ny_n.$$

Proof. Let $X=(x_1,x_2,\cdots,x_n)\in R_+^n$ and $f(X)=x_1+x_2+\cdots+x_n$. Then matrix Ω in Theorem 2.1 is

$$\Omega=\begin{pmatrix} \frac{f}{x_1}-1 & -1 & \cdots & -1 \\ -1 & \frac{f}{x_2}-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & & \frac{f}{x_n}-1\end{pmatrix}.$$

The kth-order principal submatrixes of Ω is

$$\det\begin{pmatrix} \frac{f}{x_1}-1 & -1 & \cdots & -1 \\ -1 & \frac{f}{x_2}-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{f}{x_k}-1 \end{pmatrix} = \det\begin{pmatrix} \frac{f}{x_1}-1 & -1 & \cdots & -1 \\ -\frac{f}{x_1} & \frac{f}{x_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{f}{x_1} & 0 & \cdots & \frac{f}{x_k} \end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{f}{x_1}-\frac{x_2}{x_1}-\cdots-\frac{x_k}{x_1}-1 & -1 & \cdots & -1 \\ 0 & \frac{f}{x_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{f}{x_k} \end{pmatrix} = \det\begin{pmatrix} \frac{x_{k+1}+\cdots+x_n}{x_1} & -1 & \cdots & -1 \\ 0 & \frac{f}{x_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{f}{x_k} \end{pmatrix} \geq 0.$$

So f is a geometrically convex function according to Theorem 2.1. Further for $X=(x_1,x_2,\cdots,x_n)\in R_+^n$, $Y=(y_1,y_2,\cdots,y_n)\in R_+^n$, we have

$$(f(X^p))^{\frac{1}{p}}\cdot(f(Y^q))^{\frac{1}{q}} \geq f\left(\left(X^P\right)^{\frac{1}{p}}\cdot(Y^q)^{\frac{1}{q}}\right) = f(X\cdot Y),$$

$$(x_1^p+x_2^p+\cdots+x_n^p)^{\frac{1}{p}}(y_1^q+y_2^q+\cdots+y_n^q)^{\frac{1}{q}} \geq x_1y_1+x_2y_2+\cdots+x_ny_n,$$

where $p>1, q>1$, $\frac{1}{p}+\frac{1}{q}=1$. ∎

Example 3.2. *Let $p>1, q>1, \frac{1}{p}+\frac{1}{q}=1, a_1, a_2, b_1, b_2>0$. The mean with one-parameter of a and b is defined by*

$$J_r(a,b)=\begin{cases} \frac{r\left(b^{r+1}-a^{r+1}\right)}{(r+1)(b^r-a^r)}, & r\neq 0,-1, a\neq b, \\ \frac{b-a}{\ln b-\ln a}, & r=0, a\neq b, \\ \frac{ab(\ln b-\ln a)}{b-a}, & r=-1, b\neq a, \\ a, & a=b. \end{cases}$$

Then

$$J_r(a_1b_1,a_2b_2)\leq [J_r(a_1^p,a_2^p)]^{\frac{1}{p}}\cdot[J_r(b_1^q,b_2^q)]^{\frac{1}{q}} \tag{3.1}$$

if $r>-\frac{1}{2}$,

$$J_r(a_1b_1,a_2b_2)\geq [J_r(a_1^p,a_2^p)]^{\frac{1}{p}}\cdot[J_r(b_1^q,b_2^q)]^{\frac{1}{q}} \tag{3.2}$$

if $r<-\frac{1}{2}$.

Proof. If $r\neq 0,-1$, it is easy to prove that $\frac{\partial F_r}{\partial a}, \frac{\partial F_r}{\partial b}, \frac{\partial^2 F_r}{\partial a\partial b}, \frac{\partial^2 F_r}{\partial b\partial a}$ are existent and continuous according to definition of partial derivative. Moreover if $a\neq b$,

$$\frac{\partial J_r}{\partial a}=\frac{r}{r+1}\cdot\frac{a^{2r}-(r+1)a^rb^r+ra^{r-1}b^{r+1}}{(b^r-a^r)^2},$$

$$\frac{\partial J_r}{\partial b}=\frac{r}{r+1}\cdot\frac{b^{2r}-(r+1)a^rb^r+ra^{r+1}b^{r-1}}{(b^r-a^r)^2},$$

$$\frac{\partial^2 J_r}{\partial a\partial b}=\frac{r}{r+1}\cdot\frac{\left(r^2-r\right)a^{2r}b^{r-1}-r(r+1)a^{2r-1}b^r+r(r+1)a^rb^{2r-1}-\left(r^2-r\right)a^{r-1}b^{2r}}{(b^r-a^r)^3}$$

and

$$\frac{\partial J_r}{\partial a}\cdot\frac{\partial J_r}{\partial b}-\frac{\partial^2 J_r}{\partial a\partial b}$$

$$=\left(\frac{r}{r+1}\right)^2\cdot\frac{r^2\left(a^{3r+1}b^{r-1}+a^{r-1}b^{3r+1}\right)-\left(r^2+2r+1\right)\left(a^{3r}b^r+a^rb^{3r}\right)a^{3r}b^r+(4r+2)a^{2r}b^{2r}}{(b^r-a^r)^4}$$

$$= \frac{r^2 a^{r-1} b^{r-1}}{(b^r - a^r)^2} \cdot \left[\left(\frac{r\left(b^{r+1} - a^{r+1}\right)}{(r+1)(b^r - a^r)} \right)^2 - ab \right] = \frac{r^2 a^{r-1} b^{r-1}}{(b^r - a^r)^2} \cdot \left[(J_r(a,b))^2 - \left(J_{-\frac{1}{2}}(a,b) \right)^2 \right].$$

Because J_r is a monotone increasing function [12] with respect to r, $J_r(a,b) > J_{-\frac{1}{2}}(a,b) = \sqrt{ab}$, $\frac{\partial J_r}{\partial a} \cdot \frac{\partial J_r}{\partial b} - \frac{\partial^2 J_r}{\partial a \partial b} \geq 0$, where $r > -\frac{1}{2}$. So J_r is a geometrically convex function according to Theorem 2.2 with $r > -\frac{1}{2}$. Further according to the definition of geometrically convex functions, inequality (3.1) holds. Similarly if $r < -\frac{1}{2}$, J_r is a geometrically concave function, inequality (3.2) holds. If $r = 0, -1$, (3.1) and (3.2) hold because of continuity of J_r with respect to r. ∎

Example 3.3. *Let $a, b \in R_+$, $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$ respectively the arithmetic and geometric means of a and b. The logarithmic and identric means of a and b are defined by*

$$L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases}$$

and

$$I(a,b) = \begin{cases} \frac{1}{e} b^{\frac{b}{b-a}} a^{\frac{a}{a-b}}, & a \neq b, \\ a, & a = b. \end{cases}$$

Suppose $f(a,b) = \frac{(L(a,b))^2}{I(a,b)}$, $(a,b) \in R_+^2$. Then f is a geometrically convex functions.

Proof. Let $a \neq b$. It is easy to prove

$$(\ln f)_1' = \frac{1}{a-b} + \frac{2}{a(\ln b - \ln a)} + \frac{b \ln a}{(a-b)^2} - \frac{b \ln b}{(a-b)^2},$$

$$(\ln f)_{12}'' = \frac{2}{(a-b)^2} \left[\frac{A(a,b)}{L(a,b)} - \frac{L^2(a,b)}{G^2(a,b)} \right].$$

Because $L^3(a,b) \geq G^2(a,b) \cdot A(a,b)$ [11], $(\ln f)_{13}'' \leq 0$ with $a \neq b$. It is also easy to prove $(\ln f)_{12}'' \leq 0$ with $a = b$. So f is a geometrically convex function. ∎

Corollary 3.1. *Let $a, b > 0$. Then $L^2(a,b) \geq G(a,b) \cdot I(a,b)$.*

Proof. According to example 3.3 and inequality 1.5,

$$\frac{L^2(a,b)}{I(a,b)} = \sqrt{\frac{L^2(a,b)}{I(a,b)} \cdot \frac{L^2(b,a)}{I(b,a)}} \geq \frac{L^2\left(\sqrt{ab}, \sqrt{ab}\right)}{I\left(\sqrt{ab}, \sqrt{ab}\right)} = \sqrt{ab} = G(a,b).$$

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