

**GENERALIZATIONS OF WEIGHTED OSTROWSKI TYPE
INEQUALITIES FOR MAPPINGS OF BOUNDED VARIATION
AND THEIR APPLICATIONS**

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ABSTRACT. In this paper, we establish some generalizations of weighted Ostrowski type Inequalities, and give several applications for r -moments, expectation of a continuous random variable and the Beta mapping.

1. INTRODUCTION

Throughout this section, let $a < b$ in \mathbb{R} , $I_n : a = x_0 < x_1 < \dots < x_n = b$ be a partition of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$), $l_i := x_{i+1} - x_i$ ($i = 0, 1, \dots, n-1$) and $\nu(l) = \max_{i=0,1,\dots,n-1} l_i$.

The *Ostrowski's inequality* [10, p. 469], states that if f' exists and is bounded on (a, b) , then, for all $x \in [a, b]$, we have the inequality

$$(1.1) \quad \left| \int_a^b f(t)dt - f(x)(b-a) \right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty,$$

where

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty.$$

Now if f is as above, then we can approximate the integral $\int_a^b f(t) dt$ by the *Ostrowski quadrature formula* $A_O(f, I_n, \xi)$, having an error given by $R_O(f, I_n, \xi)$, where

$$A_O(f, I_n, \xi) := \sum_{i=1}^n f(\xi_i) l_i,$$

and the remainder satisfies the estimation

$$|R_O(f, I_n, \xi)| \leq \sum_{i=0}^{n-1} \left[\frac{1}{4} l_i^2 + \left(\xi_i - \frac{x_{i-1} + x_i}{2} \right)^2 \right] \|f'\|_\infty.$$

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2, 3, 8, 9].

Recently, Dragomir [2] proved the following two Ostrowski type inequalities for mappings of bounded variation:

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Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation. Then

$$(1.2) \quad \left| \int_a^b f(t)dt - f(x)(b-a) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f)$$

for all $x \in [a, b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. The constant $\frac{1}{2}$ is the best possible.

Theorem 2. Let $A_O(f, I_n, \xi)$ and $R_O(f, I_n, \xi)$ be as above and let f and $\bigvee_a^b(f)$ be defined as in Theorem 1, then we have

$$\int_a^b f(t)dt = A_O(f, I_n, \xi) + R_O(f, I_n, \xi),$$

and the remainder term $R_O(f, I_n, \xi)$ satisfies the estimation

$$(1.3) \quad \begin{aligned} |R_O(f, I_n, \xi)| &\leq \max_{i=0,1,\dots,n-1} \left[\frac{1}{2}l_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b(f) \\ &\leq \left[\frac{1}{2}\nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b(f) \\ &\leq \nu(l) \bigvee_a^b(f). \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in (1.3).

The *Simpson's inequality*, states that if $f^{(4)}$ exists and is bounded on (a, b) , then

$$(1.4) \quad \left| \int_a^b f(t)dt - \frac{b-a}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{(b-a)^5}{2880} \|f^{(4)}\|_\infty,$$

where

$$\|f^{(4)}\|_\infty := \sup_{t \in (a,b)} |f^{(4)}(t)| < \infty.$$

Let f be as above, then we can approximate the integral $\int_a^b f(t)dt$ by the *Simpson's quadrature formula* $A_S(f, I_n)$, having an error given by $R_S(f, I_n)$, where

$$A_S(f, I_n) := \sum_{i=0}^{n-1} \frac{l_i}{3} \left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(\frac{x_i + x_{i+1}}{2}\right) \right],$$

and the remainder satisfies the estimation

$$|R_S(f, I_n)| \leq \frac{1}{2880} \|f^{(4)}\|_\infty \sum_{i=0}^{n-1} l_i^5.$$

For some recent results which generalize, improve and extend this classic inequality (1.4), see the papers [4] – [7], [12] – [14].

Recently, Dragomir [6] proved the following two Simpson type inequalities for mappings of bounded variation:

Theorem 3. Let f and $\bigvee_a^b(f)$ be defined as in Theorem 2. Then

$$(1.5) \quad \left| \int_a^b f(t)dt - \frac{b-a}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{1}{3}(b-a) \bigvee_a^b(f).$$

The constant $\frac{1}{3}$ is the best possible.

Theorem 4. Let $A_S(f, I_n)$ and $R_S(f, I_n)$ be as above and let f and $V_a^b(f)$ be defined as in Theorem 3, then we have

$$\int_a^b f(t)dt = A_S(f, I_n) + R_S(f, I_n)$$

and the remainder term $R_S(f, I_n)$ satisfies the estimation

$$(1.6) \quad |R_S(f, I_n)| \leq \frac{1}{3} \nu(l) \bigvee_a^b(f).$$

The constant $\frac{1}{3}$ is the best possible.

In this paper, we establish weighted generalizations of Theorems 1 – 4, and give several applications for r -moments, expectation of a continuous random variable and the Beta mapping.

2. SOME INTEGRAL INEQUALITIES

Theorem 5. Let $0 \leq \alpha \leq 1$, $g : [a, b] \rightarrow [0, \infty)$ be continuous and positive on (a, b) and let $h : [a, b] \rightarrow \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$. Let $c = h^{-1}\left(\left(1 - \frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$ and $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1 - \frac{\alpha}{2}\right)h(b)\right)$. Suppose that f and $V_a^b(f)$ are defined as in Theorem 4. Then, for all $x \in [c, d]$, we have

$$(2.1) \quad \left| \int_a^b f(t)g(t) dt - \left[(1 - \alpha) f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt \right| \leq K \cdot \bigvee_a^b(f),$$

where

$$K := \begin{cases} \frac{1-\alpha}{2} \int_a^b g(t) dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right|, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \max \left\{ \frac{1-\alpha}{2} \int_a^b g(t) dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right|, \frac{\alpha}{2} \int_a^b g(t) dt \right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2} \int_a^b g(t) dt, & \text{if } \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

and $V_a^b(f)$ denotes the total variation of f on the interval $[a, b]$. In (2.1), the constant $\frac{\alpha}{2}$ as $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{1-\alpha}{2}$ as $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

Proof. Let $x \in [c, d]$. Define

$$s(t) := \begin{cases} h(t) - \left[\left(1 - \frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b) \right], & t \in [a, x] \\ h(t) - \left[\frac{\alpha}{2}h(a) + \left(1 - \frac{\alpha}{2}\right)h(b) \right], & t \in [x, b] \end{cases}.$$

Using integration by parts, we have the following identity

$$\begin{aligned}
& \int_a^b s(t) df(t) \\
&= \left[h(t) - \left[\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b) \right] \right] \cdot f(t) \Big|_{t=a}^{t=x} - \int_a^x f(t)g(t) dt \\
&\quad + \left[h(t) - \left[\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b) \right] \right] \cdot f(t) \Big|_{t=x}^{t=b} - \int_x^b f(t)g(t) dt \\
&= \left[(1 - \alpha) f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] [h(b) - h(a)] - \int_a^b f(t)g(t) dt \\
(2.2) \quad &= \left[(1 - \alpha) f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt - \int_a^b f(t)g(t) dt.
\end{aligned}$$

It is well known [1, p. 159] that if $\mu, \nu : [a, b] \rightarrow \mathbb{R}$ are such that μ is continuous on $[a, b]$ and ν is of bounded variation on $[a, b]$, then $\int_a^b \mu(t) d\nu(t)$ exists and [1, p. 177]

$$(2.3) \quad \left| \int_a^b \mu(t) d\nu(t) \right| \leq \sup_{x \in [a, b]} |\mu(x)| \bigvee_a^b(\nu).$$

Now, using (2.2) and (2.3), we have

$$\begin{aligned}
(2.4) \quad & \left| \int_a^b f(t)g(t) dt - \left[(1 - \alpha) f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt \right| \\
& \leq \sup_{t \in [a, b]} |s(t)| \bigvee_a^b(f).
\end{aligned}$$

Since $h(t) - \left[\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b) \right]$ is increasing on the interval $[a, x]$, $h(t) - \left[\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b) \right]$ is increasing on the interval $[x, b]$, $\max\{\sigma, \rho\} = \frac{\sigma + \rho}{2} + \frac{1}{2} |\sigma - \rho|$ for $\sigma, \rho \in \mathbb{R}$ and

$$\left| h(x) - \frac{h(a) + h(b)}{2} \right| \leq \frac{1 - \alpha}{2} (h(b) - h(a)) = \frac{1 - \alpha}{2} \int_a^b g(t) dt,$$

we have

$$\begin{aligned}
& \sup_{t \in [a, b]} |s(t)| \\
&= \max \left\{ h(x) - \left[\left(1 - \frac{\alpha}{2}\right) h(a) + \frac{\alpha}{2} h(b) \right], \right. \\
&\quad \left. \left[\frac{\alpha}{2} h(a) + \left(1 - \frac{\alpha}{2}\right) h(b) \right] - h(x), \frac{\alpha}{2} [h(b) - h(a)] \right\} \\
&= \max \left\{ \frac{1 - \alpha}{2} [h(b) - h(a)] + \left| h(x) - \frac{h(a) + h(b)}{2} \right|, \frac{\alpha}{2} [h(b) - h(a)] \right\} \\
&= \max \left\{ \frac{1 - \alpha}{2} \int_a^b g(t) dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right|, \frac{\alpha}{2} \int_a^b g(t) dt \right\} \\
(2.5) \quad &= K.
\end{aligned}$$

Thus, by (2.4) and (2.5), we obtain (2.1).

Suppose $0 \leq \alpha \leq \frac{1}{2}$. We assume that the inequality (2.1) holds with a constant $C_1 > 0$, i.e.,

$$\begin{aligned} \left| \int_a^b f(t)g(t) dt - \left[(1-\alpha)f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t) dt \right| \\ \leq \left[C_1 \int_a^b g(t) dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right] \cdot \bigvee_a^b(f). \end{aligned}$$

Let

$$f(t) = \begin{cases} 0 & \text{as } t \in [a, b] \setminus \left\{ h^{-1} \left(\frac{h(a)+h(b)}{2} \right) \right\} \\ \frac{1}{2} & \text{as } t = h^{-1} \left(\frac{h(a)+h(b)}{2} \right) \end{cases}.$$

Then f is with bounded variation on $[a, b]$, and

$$\int_a^b f(t)g(t) dt = 0, \quad \bigvee_a^b(f) = 1$$

and for $x = h^{-1} \left(\frac{h(a)+h(b)}{2} \right)$, we get in (2.1)

$$\frac{1-\alpha}{2} \leq C_1,$$

which implies the constant $\frac{1-\alpha}{2}$ is the best possible.

Suppose $\frac{2}{3} \leq \alpha \leq 1$. We assume that the inequality (2.1) holds with a constant $C_2 > 0$, i.e.,

$$\begin{aligned} \left| \int_a^b f(t)g(t) dt - \left[(1-\alpha)f(x) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t) dt \right| \\ \leq C_2 \int_a^b g(t) dt \cdot \bigvee_a^b(f). \end{aligned}$$

Let

$$f(t) = \begin{cases} 0 & \text{as } t \in [a, b) \\ 1 & \text{as } t = b \end{cases}.$$

Then f is with bounded variation on $[a, b]$ and

$$\int_a^b f(t)g(t) dt = 0, \quad \bigvee_a^b(f) = 1,$$

we get in (2.1)

$$\frac{\alpha}{2} \leq C_2$$

which implies the constant $\frac{\alpha}{2}$ is the best possible.

This completes the proof. ■

Under the conditions of Theorem 5, we have the following remarks and corollaries.

Remark 1.

- (1) If we choose $\alpha = 0$ and $g(t) \equiv 1, h(t) = t$ on $[a, b]$, then the inequality (2.1) reduces to (1.2).
- (2) If we choose $\alpha = \frac{1}{3}, g(t) \equiv 1, h(t) = t$ on $[a, b]$ and $x = \frac{a+b}{2}$, then the inequality (2.1) reduces to (1.5).
- (3) If we choose $\alpha = 0$, then for all $x \in [a, b]$ the inequality (2.1) reduces to the following inequality

$$\left| \int_a^b f(t)g(t) dt - f(x) \cdot \int_a^b g(t) dt \right| \leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right| \right] \cdot \bigvee_a^b(f),$$

which is the “weighted Ostrowski” inequality.

- (4) If we choose $\alpha = 1$, then the inequality (2.1) reduces to the following inequality

$$\left| \int_a^b f(t)g(t) dt - \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \right| \leq \frac{1}{2} \int_a^b g(t) dt \cdot \bigvee_a^b(f)$$

which is the “weighted trapezoid” inequality.

- (5) If we choose $\alpha = \frac{1}{3}$ and $x = h^{-1}\left(\frac{h(a)+h(b)}{2}\right)$, then the inequality (2.1) reduces to the following inequality

$$\left| \int_a^b f(t)g(t) dt - \left[\frac{2}{3}f(x) + \frac{1}{3} \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt \right| \leq \frac{1}{3} \int_a^b g(t) dt \cdot \bigvee_a^b(f)$$

which is the “weighted Simpson” inequality.

Corollary 1. Let $0 \leq \alpha \leq 1, f \in C^{(1)}[a, b]$. Then we have the inequality

$$\left| \int_a^b f(t)g(t) dt - \left[(1 - \alpha)f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt \right| \leq K \cdot \|f'\|_1$$

for all $x \in [c, d]$, where $\|\cdot\|_1$ is the L_1 -norm, namely

$$\|f'\|_1 := \int_a^b |f'(t)| dt.$$

Corollary 2. Let $0 \leq \alpha \leq 1, f : [a, b] \rightarrow \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality

$$\left| \int_a^b f(t)g(t) dt - \left[(1 - \alpha)f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt \right| \leq KL(b - a)$$

for all $x \in [c, d]$.

Corollary 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic mapping. Then we have the inequality

$$\left| \int_a^b f(t)g(t) dt - \left[(1 - \alpha)f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) dt \right| \leq K \cdot |f(b) - f(a)|$$

for all $x \in [c, d]$.

Remark 2. The following inequality is well-known in the literature as the Bullen's inequality [11, p. 141]:

$$(2.6) \quad \int_a^b f(t)dt \leq \frac{b-a}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \leq (b-a) \frac{f(a)+f(b)}{2},$$

where $f : [a, b] \rightarrow \mathbb{R}$ is convex. Using the above results and (2.1), letting $\alpha = \frac{1}{2}$, $g(t) \equiv 1$ on $[a, b]$, $h(t) = t$ on $[a, b]$, $x = \frac{a+b}{2}$, we obtain the following error bound of the first inequality in (2.6),

$$0 \leq \frac{b-a}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] - \int_a^b f(t)dt \leq \frac{1}{4} (b-a) \bigvee_a^b(f),$$

provided that f is of bounded variation on $[a, b]$.

3. APPLICATIONS FOR QUADRATURE FORMULA

Throughout this section, let $a < b$ in \mathbb{R} and let α, g and h be defined as in Theorem 5. Let $f : [a, b] \rightarrow \mathbb{R}$, and let $I_n : a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$ and $c_i = h^{-1}\left(\left(1 - \frac{\alpha}{2}\right)h(x_i) + \frac{\alpha}{2}h(x_{i+1})\right)$, $d_i = h^{-1}\left(\frac{\alpha}{2}h(x_i) + \left(1 - \frac{\alpha}{2}\right)h(x_{i+1})\right)$ and $\zeta_i \in [c_i, d_i]$ ($i = 0, 1, \dots, n-1$). Put $L_i := h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t) dt$ and define the sum

$$A_O(f, g, h, I_n, \zeta) := \sum_{i=0}^{n-1} \left[(1-\alpha) f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i$$

and

$$R_O(f, g, h, I_n, \zeta) = \int_a^b f(t)g(t)dt - A_O(f, g, h, I_n, \zeta).$$

We have the following approximation of the integral $\int_a^b f(t)g(t)dt$.

Theorem 6. Let f be defined as in Theorem 5 and let

$$\int_a^b f(t)g(t)dt = A_O(f, g, h, I_n, \zeta) + R_O(f, g, h, I_n, \zeta),$$

then, the remainder term $R_O(f, g, h, I_n, \zeta)$ satisfies the estimation

$$(3.1) \quad \begin{aligned} |R_O(f, g, h, I_n, \zeta)| &\leq \sum_{i=0}^{n-1} K_i \bigvee_{x_i}^{x_{i+1}}(f) \\ &\leq M_1 \cdot \bigvee_a^b(f) \\ &\leq M_2 \cdot \bigvee_a^b(f) \\ &\leq M_3 \cdot \bigvee_a^b(f), \end{aligned}$$

where

$$K_i := \begin{cases} \frac{1-\alpha}{2}L_i + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \max \left\{ \frac{1-\alpha}{2}L_i + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, \frac{\alpha}{2}L_i \right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}L_i, & \text{if } \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

$$(i = 0, 1, \dots, n-1),$$

$$M_1 := \begin{cases} \max_{i=0,1,\dots,n-1} \left\{ \frac{1-\alpha}{2}L_i + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right| \right\}, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \max_{i=0,1,\dots,n-1} \left\{ \max \left\{ \frac{1-\alpha}{2}\nu(L) + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, \frac{\alpha}{2}\nu(L) \right\} \right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\nu(L), & \text{if } \frac{2}{3} \leq \alpha \leq 1 \end{cases},$$

$$M_2 := \begin{cases} \frac{1-\alpha}{2}\nu(L) + \max_{i=0,1,\dots,n-1} \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \max_{i=0,1,\dots,n-1} \left\{ \max \left\{ \frac{1-\alpha}{2}\nu(L) + \left| h(\zeta_i) - \frac{h(x_i)+h(x_{i+1})}{2} \right|, \frac{\alpha}{2}\nu(L) \right\} \right\}, & \text{if } \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\nu(L), & \text{if } \frac{2}{3} \leq \alpha \leq 1 \end{cases},$$

$$M_3 := \begin{cases} (1-\alpha)\nu(L), & \text{if } 0 \leq \alpha \leq \frac{2}{3} \\ \frac{\alpha}{2}\nu(L), & \text{if } \frac{2}{3} \leq \alpha \leq 1 \end{cases},$$

and $\nu(L) := \max \{L_i \mid i = 0, 1, \dots, n-1\}$. In the third inequality of (3.1), the constant $\frac{\alpha}{2}$ as $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{1-\alpha}{2}$ as $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

Proof. Apply Theorem 5 on the intervals $[x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) to get

$$\left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - \left[(1-\alpha)f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i \right| \leq K_i \bigvee_{x_i}^{x_{i+1}}(f),$$

for all $i = 0, 1, \dots, n-1$.

Using this and the generalized triangle inequality, we have

$$\begin{aligned} & |R_O(f, g, h, I_n, \zeta)| \\ & \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - \left[(1-\alpha)f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i \right| \\ & \leq \sum_{i=0}^{n-1} K_i \bigvee_{x_i}^{x_{i+1}}(f) \\ & \leq \left(\max_{i=0,1,\dots,n-1} K_i \right) \cdot \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f) = M_1 \cdot \bigvee_a^b(f) \leq M_2 \cdot \bigvee_a^b(f) \end{aligned}$$

and the first inequality, second inequality and third inequality in (3.1) are proved.

For the fourth inequality in (3.1), we observe that

$$\left| h(\zeta_i) - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1-\alpha}{2} \cdot L_i \quad (i = 0, 1, \dots, n-1);$$

and then

$$\max_{i=0,1,\dots,n-1} \left| h(\zeta_i) - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1-\alpha}{2} \nu(L)$$

and $M_2 \leq M_3$. Thus the theorem is proved. ■

Under the conditions of Theorem 6, we have the following remarks and corollaries.

Remark 3.

- (1) If we choose $\alpha = 0$ and $g(t) \equiv 1, h(t) = t$ on $[a, b]$ and $\xi_i = \zeta_i$ ($i = 0, 1, \dots, n-1$), then the inequality (3.1) reduces to (1.3).
- (2) If we choose $\alpha = \frac{1}{3}, g(t) \equiv 1, h(t) = t$ on $[a, b]$ and $\zeta_i = \frac{x_i + x_{i+1}}{2}$ ($i = 0, 1, \dots, n-1$), then the third inequality in (3.1) reduces to (1.6).

Corollary 4. In Theorem 6, let $f : [a, b] \rightarrow \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$ and choose $\zeta_i := h^{-1} \left(\frac{h(x_i) + h(x_{i+1})}{2} \right)$ ($i = 0, 1, \dots, n-1$). Then

$$M_1 := \begin{cases} \frac{(1-\alpha)}{2} \nu(L), & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha}{2} \nu(L), & \text{if } \frac{1}{2} \leq \alpha \leq 1 \end{cases}$$

and we have the formula

$$\begin{aligned} \int_a^b f(t)g(t)dt &= A_O(f, g, h, I_n, \zeta) + R_O(f, g, h, I_n, \zeta) \\ &= \sum_{i=0}^{n-1} \left[(1-\alpha) f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i + R_O(f, g, h, I_n, \zeta) \end{aligned}$$

and the remainder satisfies the estimation

$$|R_O(f, g, h, I_n, \zeta)| \leq M_1 L (b-a).$$

Corollary 5. In Theorem 6, let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic mapping and let ζ_i ($i = 0, 1, \dots, n-1$) and M_1 be defined as in Corollary 4. Then the remainder $R_O(f, g, h, I_n, \zeta)$ satisfies the estimation

$$|R_O(f, g, h, I_n, \zeta)| \leq M_1 \cdot |f(b) - f(a)|.$$

The case of equidistant divisions is embodied in the following corollary and remark:

Corollary 6. Suppose that

$$x_i := h^{-1} \left[h(a) + \frac{i(h(b) - h(a))}{n} \right] \quad (i = 0, 1, \dots, n)$$

and

$$\begin{aligned} L_i &:= h(x_{i+1}) - h(x_i) \\ &= \frac{h(b) - h(a)}{n} = \frac{1}{n} \int_a^b g(t) dt \quad (i = 0, 1, \dots, n-1). \end{aligned}$$

In Theorem 6, let $\zeta_i = h^{-1} \left(\frac{h(x_i) + h(x_{i+1})}{2} \right)$ ($i = 0, 1, \dots, n-1$), then

$$M_1 := \begin{cases} \frac{(1-\alpha)}{2n} \int_a^b g(t) dt, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha}{2n} \int_a^b g(t) dt, & \text{if } \frac{1}{2} \leq \alpha \leq 1 \end{cases}$$

and we have the formula

$$\begin{aligned} \int_a^b f(t)g(t)dt &= A_O(f, g, h, I_n, \zeta) + R_O(f, g, h, I_n, \zeta) \\ &= \frac{1}{n} \int_a^b g(t) dt \cdot \sum_{i=0}^{n-1} \left[(1-\alpha) f(\zeta_i) + \alpha \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right] L_i \\ &\quad + R_O(f, g, h, I_n, \zeta) \end{aligned}$$

and the remainder satisfies the estimate

$$|R_O(f, g, h, I_n, \zeta)| \leq M_1 \cdot \bigvee_a^b(f).$$

Remark 4. If we want to approximate the integral $\int_a^b f(t)g(t)dt$ by $A_O(f, g, h, I_n, \zeta)$ with an accuracy less than $\varepsilon > 0$, we need at least $n_\varepsilon \in \mathbb{N}$ points for the partition I_n , where

$$K_\varepsilon := \begin{cases} \frac{(1-\alpha)}{2\varepsilon} \int_a^b g(t) dt, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha}{2\varepsilon} \int_a^b g(t) dt, & \text{if } \frac{1}{2} \leq \alpha \leq 1 \end{cases}, \quad n_\varepsilon := \left\lceil K_\varepsilon \cdot \bigvee_a^b(f) \right\rceil + 1$$

and $[r]$ denotes the Gaussian integer of $r \in \mathbb{R}$.

4. SOME INEQUALITIES FOR RANDOM VARIABLES

Throughout this section, let $0 < a < b$ in \mathbb{R} , $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density function $g : [a, b] \rightarrow [0, \infty)$ which is positive on (a, b) and assume that the r -moment

$$E_r(X) := \int_a^b t^r g(t) dt,$$

is finite.

Theorem 7. *The inequality*

$$(4.1) \quad \left| E_r(X) - \left[(1-\alpha) \cdot \left(h^{-1} \left(\frac{1}{2} \right) \right)^r + \alpha \cdot \frac{a^r + b^r}{2} \right] \right| \leq \bar{K} \cdot |b^r - a^r|$$

holds where $h(t) = \int_a^t g(x) dx$ ($t \in [a, b]$) and

$$\bar{K} := \begin{cases} \frac{(1-\alpha)}{2}, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha}{2}, & \text{if } \frac{1}{2} \leq \alpha \leq 1 \end{cases}.$$

Proof. If we put $f(t) = t^r$, and $x = h^{-1} \left(\frac{h(a)+h(b)}{2} \right)$ in Corollary 3, then

$$\bar{K} = K = \begin{cases} \frac{(1-\alpha)}{2}, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha}{2}, & \text{if } \frac{1}{2} \leq \alpha \leq 1 \end{cases}$$

and we obtain the inequality

$$(4.2) \quad \left| \int_a^b f(t)g(t) dt - \left[(1-\alpha) \cdot f \left(h^{-1} \left(\frac{h(a)+h(b)}{2} \right) \right) + \alpha \cdot \frac{f(a)+f(b)}{2} \right] \int_a^b g(t) dt \right| \leq \overline{K} \cdot |f(b) - f(a)|.$$

Since

$$\int_a^b f(t)g(t) dt = E_r(X), \quad h(a) = 0, \quad h(b) = \int_a^b g(t) dt = 1, \\ \frac{f(a)+f(b)}{2} = \frac{a^r+b^r}{2}, \quad \text{and} \quad |f(b) - f(a)| = |b^r - a^r|,$$

(4.1) follows from (4.2). ■

If we choose $r = 1$ in Theorem 7, then we have the following remark:

Remark 5. *If $E(X)$ is the expectation of the random variable X , then*

$$\left| E(X) - \left[(1-\alpha) \cdot h^{-1} \left(\frac{1}{2} \right) + \alpha \cdot \frac{a+b}{2} \right] \right| \leq \overline{K} \cdot (b-a).$$

5. AN INEQUALITY FOR THE BETA MAPPING

The following mapping is well-known in the literature as the *Beta mapping*:

$$\beta(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p > 0, q > 0.$$

Theorem 8. *Let $p > 0$, $q > 1$ and n be a positive integer. Then the inequality*

$$(5.1) \quad \left| \beta(p, q) - \frac{1}{np} \sum_{i=0}^{n-1} \left\{ \frac{\alpha}{2} \left(\left[1 - \left(\frac{i}{n} \right)^{\frac{1}{p}} \right]^{q-1} + \left[1 - \left(\frac{i+1}{n} \right)^{\frac{1}{p}} \right]^{q-1} \right) + (1-\alpha) \left[1 - \left(\frac{2i+1}{2n} \right)^{\frac{1}{p}} \right]^{q-1} \right\} \right| \leq \overline{M}$$

holds where

$$\overline{M} := \begin{cases} \frac{(1-\alpha)}{2np}, & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha}{2np}, & \text{if } \frac{1}{2} \leq \alpha \leq 1 \end{cases}.$$

Proof. If we put $a = 0$, $b = 1$, $f(t) = (1-t)^{q-1}$, $g(t) = t^{p-1}$ and $h(t) = \frac{t^p}{p}$ ($t \in [0, 1]$) in Corollary 6, then, $\int_a^b g(t)dt = \frac{1}{p}$, $h^{-1}(t) = (pt)^{\frac{1}{p}}$ ($t \in [0, 1]$), $x_i = \left(\frac{i}{n}\right)^{\frac{1}{p}}$ ($i = 0, 1, \dots, n$), $\zeta_i = \frac{2i+1}{2np}$ ($i = 0, 1, \dots, n-1$), $\bigvee_a^b(f) = 1$ and $\overline{M} = M_1$, so that the inequality (5.1) holds. ■

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