

UNIVERSAL MULTIPLICATION TABLE

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ABSTRACT. In this paper, considering the concept of Universal Multiplication Table, we show that for every $n \geq 2$, the inequality:

$$M(n) = \#\{ij | 1 \leq i, j \leq n\} \geq \frac{n^2}{\mathfrak{N}(n^2)},$$

holds true with:

$$\mathfrak{N}(n) = n^{\frac{\log 2}{\log \log n}} \left(1 + \frac{387}{200 \log \log n} \right).$$

1. INTRODUCTION

Consider the following $n \times n$ *Multiplication Table*, which we denote it by $MT_{n \times n}$:

1	2	3	...	n
2	4	6	...	$2n$
3	6	9	...	$3n$
\vdots	\vdots	\vdots	\ddots	\vdots
n	$2n$	$3n$...	n^2

Let $\mathfrak{M}(n; k)$ be the number of k 's, which appear in $MT_{n \times n}$; i.e.

$$(1.1) \quad \mathfrak{M}(n; k) = \#\{(a, b) \in \mathbb{N}_n^2 \mid ab = k\},$$

where $\mathbb{N}_n = \mathbb{N} \cap [1, n]$. For example, we have:

$$\mathfrak{M}(2; 2) = 2, \mathfrak{M}(7; 6) = 4, \mathfrak{M}(10; 9) = 3, \mathfrak{M}(100; 810) = 10, \mathfrak{M}(100; 9900) = 2.$$

In this paper first we study some elementary properties of the function $\mathfrak{M}(n; k)$, for a fixed $n \in \mathbb{N}$. Then we try to connect $\mathfrak{M}(n; k)$ by the famous *Multiplication Table Function*¹; $M(n) = \#\{ij | (i, j) \in \mathbb{N}_n^2\}$ in order to get some lower bounds for it. To do this, we introduce the concept of *Universal Multiplication Table*, which is an infinite array generated by multiplying the components of points in the infinite lattice \mathbb{N}^2 . Let $D(n) = \{d : d > 0, d|n\}$. To get above mentioned bounds for the function $M(n)$, we will need some upper bounds for the *Divisor Function* $d(n) = \#D(n)$, which we recall best known, due to J.L. Nicolas [5]:

$$\frac{\log d(n)}{\log 2} \leq \frac{\log n}{\log \log n} \left(1 + \frac{1.9349 \dots}{\log \log n} \right) \quad (n \geq 3),$$

or

$$(1.2) \quad d(n) \leq \mathfrak{N}(n)$$

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¹This sequence, has been indexed in "The On-Line Encyclopedia of Integer Sequences" data base with ID A027424. Web page of above data base is:
<http://www.research.att.com/njas/sequences/index.html>

for $n \geq 3$, with

$$\mathfrak{N}(n) = n^{\frac{\log 2}{\log \log n}} \left(1 + \frac{387}{200 \log \log n}\right).$$

2. SOME ELEMENTARY PROPERTIES OF THE FUNCTION $\mathfrak{M}(n; k)$

Considering (1.1), for every $s \in \mathbb{C}$, we have:

$$(2.1) \quad \sum_{1 \leq i, j \leq n} \frac{1}{(ij)^s} = \sum_{k=1}^{n^2} \frac{\mathfrak{M}(n; k)}{k^s} = \sum_{k=1}^{\infty} \frac{\mathfrak{M}(n; k)}{k^s}.$$

The left hand side of above identity is equal to $\zeta_n^2(s)$, in which $\zeta_n(s) = \sum_{i=1}^n \frac{1}{i^s}$, and the number of summands in the right hand side of above identity, is equal to $M(n)$. Also, summing and counting all numbers in $MT_{n \times n}$, we obtain respectively:

$$\sum_{k=1}^{n^2} k \mathfrak{M}(n; k) = \left(\frac{n(n+1)}{2}\right)^2,$$

and

$$\sum_{k=1}^{n^2} \mathfrak{M}(n; k) = n^2,$$

which both of them are special cases of (2.1) for $s = -1$ and $s = 0$, respectively. To have some formulas for the function $\mathfrak{M}(n; k)$, we define *Incomplete Divisor Function* to be $d(k; x) = \#D(k) \cap [1, x]$. This function has some properties, which we list some of them:

1. It is trivial that for every $x \geq 1$ we have:

$$1 \leq d(k; x) \leq \min\{x, d(k)\}.$$

So, $d(k; x) = O(x)$ and naturally we ask: What is the exact order of $d(k; x)$? The next property, maybe useful to find answer.

2. If we let $D(k) = \{1 = d_1, d_2, \dots, d_{d(k)} = k\}$, then we have:

$$\begin{aligned} \int_1^k d(k; x) dx &= \sum_{i=1}^{d(k)-1} (d_{i+1} - d_i) i = \sum_{i=1}^{d(k)-1} (i+1) d_{i+1} - i d_i - \sum_{i=1}^{d(k)-1} d_{i+1} \\ &= d(k) d_{d(k)} - 1 d_1 - \sum_{d|k, d>1} d = kd(k) - \sigma(k), \end{aligned}$$

where $\sigma(k) = \sum_{a \in D(k)} a$, and we have the following bound due to G. Robin [7]:

$$(2.2) \quad \sigma(n) < \mathfrak{N}(n) \quad (n \geq 3),$$

with

$$\mathfrak{N}(n) = e^\gamma n \log \log n + \frac{3241n}{5000 \log \log n},$$

where $\gamma \approx 0.5772156649$ is Euler's constant. Considering (1.2) and (2.2), we obtain the following inequality for every $k \geq 3$:

$$2k - \mathfrak{N}(k) < \int_1^k d(k; x) dx < k\mathfrak{N}(k) - k - 1.$$

In general, every knowledge about $d(k; x)$ is useful, because:

Proposition 2.1. *For every positive integers k and n , we have:*

$$\mathfrak{M}(n; k) = d(k; n) - d\left(k; \frac{k}{n}\right) + R(n; k),$$

where

$$R(n; k) = \left\lfloor \frac{k}{n} \right\rfloor - \left\lfloor \frac{k-1}{n} \right\rfloor = \begin{cases} 1, & n \mid k, \\ 0, & \text{other wise.} \end{cases}$$

Proof. Considering (1.1), we have:

$$\mathfrak{M}(n; k) = \# \{(a, b) \in \mathbb{N}_n^2 \mid ab = k\} = \sum_{d \mid k, d \leq n, \frac{k}{d} \leq n} 1 = \sum_{d \mid k, \frac{k}{n} \leq d \leq n} 1.$$

Applying the definition of $d(k; x)$, completes the proof. \square

3. UNIVERSAL MULTIPLICATION TABLE FUNCTION

We define the *Universal Multiplication Table Function* $\mathfrak{M}(k)$ to be the number of k 's, which appear in the universal multiplication table.

Proposition 3.1. *For every positive integer k , we have:*

$$\mathfrak{M}(k) = d(k).$$

Proof. Here we have two proofs:

Elementary Method. Considering the definition of universal multiplication table, we have:

$$\mathfrak{M}(k) = \lim_{n \rightarrow \infty} \mathfrak{M}(n; k) = \lim_{n \rightarrow \infty} \sum_{d \mid k, \frac{k}{n} \leq d \leq n} 1 = \sum_{d \mid k, 0 < d < \infty} 1 = d(k).$$

Analytic Method. Considering (2.1) for $\Re(s) > 1$ and taking limit both sides of it, when n tends to infinity, we obtain:

$$(3.1) \quad \sum_{k=1}^{\infty} \frac{\mathfrak{M}(k)}{k^s} = \zeta^2(s),$$

in which $\zeta(s)$ is the Riemann zeta-function. According to the Theorem 11.17 of [1], we obtain:

$$\mathfrak{M}(k) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \zeta^2(\sigma + it) k^{\sigma + it} dt, \quad (\sigma > 1).$$

Since $\zeta^2(s) = \sum_{m=1}^{\infty} d(m) m^{-s}$, we have:

$$\begin{aligned} \mathfrak{M}(k) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \zeta^2(\sigma + it) k^{\sigma + it} dt \\ &= \sum_{m=1}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\frac{k}{m}\right)^{it} dt \\ &= \sum_{m=1, m \neq k}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\frac{k}{m}\right)^{it} dt + d(k) \\ &= \sum_{m=1, m \neq k}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim_{T \rightarrow \infty} \frac{1}{T} \sin \left(T \log \left(\frac{k}{m} \right) \right) + d(k) = d(k). \end{aligned}$$

This completes the proof. \square

Now, fix positive integer k and consider $\mathfrak{M}(n; k)$, as an arithmetic function of the variable n . Clearly, $\mathfrak{M}(n; k)$ is increasing, and for $n > k$, we have $\mathfrak{M}(n; k) = \mathfrak{M}(k)$. Thus considering Proposition 3.1, we obtain:

$$(3.2) \quad \mathfrak{M}(n; k) \leq d(k),$$

and if $k \geq 3$, considering (1.2) yields that:

$$\mathfrak{M}(n; k) \leq \mathfrak{N}(k).$$

4. STATISTICAL STUDY OF $\mathfrak{M}(n; k)$ 'S

Consider $S = [\mathfrak{M}(n; k) \mid 1 \leq k \leq n^2]$ as a list of statistical data and suppose $\overline{\mathfrak{M}}(n)$ is the average of above list, then we have:

$$\overline{\mathfrak{M}}(n) = \frac{\sum_{k=1}^{n^2} \mathfrak{M}(n; k)}{\#\{ij \mid (i, j) \in \mathbb{N}_n^2\}} = \frac{n^2}{M(n)}.$$

Thus, we have:

$$(4.1) \quad M(n) = \frac{n^2}{\overline{\mathfrak{M}}(n)}.$$

Considering (3.2), it is clear that:

$$\overline{\mathfrak{M}}(n) \leq \max\{\mathfrak{M}(n; k)\}_{k=1}^{n^2} \leq \max\{d(k)\}_{k=1}^{n^2}.$$

To use (1.2), we observe that the function $\mathfrak{N}(n)$ is increasing for $n \geq 114$. So, we have:

$$\overline{\mathfrak{M}}(n) \leq \max\{d(1), d(2), \dots, d(114), d(n^2)\} \leq \max\{12, \mathfrak{N}(n^2)\} \quad (n \geq \sqrt{3}),$$

and since $\mathfrak{N}(n) > 114.1$ holds for every $n > 0$, we obtain:

$$\overline{\mathfrak{M}}(n) \leq \mathfrak{N}(n^2) \quad (n \geq 2).$$

Therefore, we have proved the following result.

Theorem 4.1. *For every $n \geq 2$, we have:*

$$M(n) \geq \frac{n^2}{\mathfrak{N}(n^2)}.$$

Remark 4.2. One of the wonderful results about $MT_{n \times n}$ is *Erdős Multiplication Table Theorem* [6], which asserts:

$$\lim_{n \rightarrow \infty} \frac{M(n)}{n^2} = 0.$$

Above theorem yields that in the Erdős's theorem, however the ratio $\frac{M(n)}{n^2}$ tends to zero, but it doesn't faster than $\frac{1}{\mathfrak{N}(n^2)}$. More precisely, Erdős showed that $M(n) = n^2(\log n)^{-c+o(1)}$ for $c = 1 + \frac{\log \log 2}{\log 2}$ [2, 3]. The following table includes some computational results about $M(n)$ by the Maple software.

n	$M(n)$	$M(n)/n^2 \approx$	n	$M(n)$	$M(n)/n^2 \approx$
10	42	0.4200000000	2000	959759	0.2399397500
50	800	0.3200000000	3000	2121063	0.2356736667
100	2906	0.2906000000	4000	3723723	0.2327326875
1000	248083	0.2480830000	5000	5770205	0.2308082000

Note that, the true order of $M(n)$ is $n^2(\log n)^{-c}(\log \log n)^{-3/2}$ [3].

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