INEQUALITIES FOR THE ZEROS OF A POLYNOMIAL

J. L. DÍAZ-BARRERO AND J. J. EGOZCUE

Abstract. An explicit ring shaped region containing all the zeros of a polynomial involving binomial coefficients and Pell numbers is obtained and some classical bounds are improved.

1. Introduction

The problem of finding bounds for the moduli of the zeros of a polynomial has occupied a central role in the theory of equations since the time of Gauss and Cauchy. Since the beginning a lot of papers devoted to give new explicit bounds or improving the known ones have been written ([1],[2], [3],[4], [5],[6]). Nevertheless, sequences of positive integers such as Fibonacci, Lucas and Pell numbers, seldom have appeared as part of close expressions of bounds ([7]).

Using an identity of Cesaro type involving Pell numbers, we determine an annulus in the complex plane containing all the zeros of a polynomial with complex coefficients. An example shows a case in which the classical bounds for the zeros are improved.

2. Main results

In the sequel, one identity of Cesaro type involving Pell numbers is obtained. This result is applied to obtain the inner and outer radii of a ring shaped region where lie all the zeros of a given polynomial with complex coefficients. We begin with the following lemma.

Lemma 2.1. *Let n be a nonnegative integer. Then,*

$$\sum_{k=0}^{n} \binom{n}{k} 5^k 2^{n-k} P_k = P_{3n}, \tag{2.1}$$

where P_n is the n^{th} Pell number defined by $P_0 = 0, P_1 = 1$ and for all $n \geq 2$, $P_n = 2P_{n-1} + P_{n-2}$.

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Proof. Since the characteristic equation of $P_n = 2P_{n-1} + P_{n-2}, P_0 = 0, P_1 = 1$ is

$$x^2 - 2x - 1 = 0, \tag{2.2}$$

then

$$P_n = \frac{1}{2\sqrt{2}}\,(a^n - b^n)$$

where $a = 1+\sqrt{2}$ and $b = 1-\sqrt{2}$ are the roots of (2.2). Therefore, taking into account that $a^3 = 5a+2$ and $b^3 = 5b+2$, we have

$$\begin{aligned} P_{3n} &= \frac{1}{2\sqrt{2}}\left(a^{3n} - b^{3n}\right) = \frac{1}{2\sqrt{2}}\Big[(5a+2)^n - (5b+2)^n\Big] \\ &= \frac{1}{2\sqrt{2}}\left[\sum_{k=0}^{n}\binom{n}{k}5^k2^{n-k}a^k - \sum_{k=0}^{n}\binom{n}{k}5^k2^{n-k}b^k\right] \\ &= \sum_{k=0}^{n}\binom{n}{k}5^k2^{n-k}\left(\frac{a^k-b^k}{2\sqrt{2}}\right) = \sum_{k=0}^{n}\binom{n}{k}5^k2^{n-k}P_k \end{aligned}$$

and (2.1) is proved. □

Next we apply the preceding result to obtain the following theorem.

Theorem 2.2. *Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial with nonzero complex coefficients. Then, all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$, where*

$$r_1 = \frac{5}{2}\min_{1\le k\le n}\sqrt[k]{\frac{2^nP_k\binom{n}{k}}{P_{3n}}\left|\frac{a_0}{a_k}\right|} \tag{2.3}$$

and

$$r_2 = \frac{2}{5}\max_{1\le k\le n}\sqrt[k]{\frac{P_{3n}}{2^n\binom{n}{k}P_k}\left|\frac{a_{n-k}}{a_n}\right|} \tag{2.4}$$

Proof. Applying the method of Cauchy, if we assume that $|z| < r_1$ then from $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \ne 0)$, we have

$$|A(z)| = \left|\sum_{k=0}^{n} a_k z^k\right| \ge |a_0| - \sum_{k=1}^{n}|a_k||z|^k > |a_0| - \sum_{k=1}^{n}|a_k|r_1^k$$

$$= |a_0| \left(1 - \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right| r_1^k\right). \tag{2.5}$$

From (2.3), we get

$$\left|\frac{a_k}{a_0}\right| r_1^k \leq \left(\frac{5}{2}\right)^k \frac{2^n \binom{n}{k} P_k}{P_{3n}}, \; k = 1, 2, \cdots, n. \tag{2.6}$$

Substituting (2.6) into (2.5) and taking into account the previous lemma, we have

$$|A(z)| > |a_0| \left(1 - \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right| r_1^k\right) \geq |a_0| \left(1 - \sum_{k=1}^{n} \left(\frac{5}{2}\right)^k \frac{2^n \binom{n}{k} P_k}{P_{3n}}\right) = 0.$$

Therefore, $A(z)$ does not have zeros in the disk $\{z \in \mathcal{C} \, : \, |z| < r_1\}$.

A well known classical result of Cauchy on location of the zeros ([8], [2]) states that all the zeros of $A(z)$ have modulus less than or equal to the unique positive root of the equation

$$B(z) = |a_n| z^n - |a_{n-1}| z^{n-1} - \cdots - |a_1| z - |a_0| = 0.$$

Hence, the second part of our statement will be proven if we show that $B(r_2) \geq 0$. From (2.4) and applying the lemma again, we get

$$\left|\frac{a_{n-k}}{a_n}\right| \leq \left(\frac{5}{2}\right)^k \frac{2^n \binom{n}{k} P_k}{P_{3n}} r_2^k, \; , k = 1, 2, \cdots, n.$$

Then,

$$\begin{aligned} B(r_2) &= |a_n| \left[r_2^n - \sum_{k=1}^{n} \left|\frac{a_{n-k}}{a_n}\right| r_2^{n-k}\right] \\ &\geq |a_n| \left[r_2^n - \sum_{k=1}^{n} \left(\left(\frac{5}{2}\right)^k \frac{2^n \binom{n}{k} P_k}{P_{3n}} r_2^k\right) r_2^{n-k}\right] \\ &= |a_n| r_2^n \left(1 - \sum_{k=1}^{n} \left(\frac{5}{2}\right)^k \frac{2^n \binom{n}{k} P_k}{P_{3n}}\right) = 0. \end{aligned}$$

Notice that in the preceding inequalities we have used the fact that

$$\sum_{k=1}^{n} \left(\frac{5}{2}\right)^k \frac{2^n \binom{n}{k} P_k}{P_{3n}} = 1$$

as follows immediately from (2.1) and this completes the proof. □

As an example, if we consider the polynomial $A(z) = z^3 + 0.01z^2 + 0.02z + 0.7$, then we have that all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where $r_1 \simeq 0.76$ and $r_2 \simeq 1.03$ improving the classical explicit bounds of Cauchy [8] $0.41 \leq |z| < 1.7$.

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Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, 08034 Barcelona. Spain

E-mail address: `jose.luis.diaz@upc.edu & juan.jose.egozcue@upc.edu`