TWO-PARAMETER GENERALIZED WEIGHTED FUNCTIONAL MEAN

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Abstract. In this paper, a generalized weighted functional mean is defined. This includes as special cases various generalizations of the two-parameters means. Some elementary properties are listed. An explicit form is given for the special case when all variables have the special weights.

1. Introduction

The generalized weighted means of the function f with weight p and two parameters r and s are defined in [1] by

$$M_{r,s}(f;p;u,v)=\begin{cases}\left[\dfrac{\int_u^v p(x)f^r(x)dx}{\int_u^v p(x)f^s(x)dx}\right]^{\frac{1}{r-s}}, & (r-s)(u-v)\neq 0;\\ \exp\left\{\dfrac{\int_u^v p(x)f^r(x)\ln f(x)dx}{\int_u^v p(x)f^r(x)dx}\right\}, & r=s, u-v\neq 0;\\ f(x), & r=s, u=v;\end{cases} \tag{1.1}$$

where $u,v,r,s\in\mathbb{R}$, $p\geq 0, f>0$ integrable functions on the interval $[u,v]\subset\mathbb{R}$.

The basic properties of $M_{r,s}(f;p;u,v)$ were studied in [2]-[9].

In this paper, we will define a generalized weighted functional mean for two parameters and prove its monotonicity. An explicit form is given for the special case when all variables have the special weights.

2. Definition And Properties

Throughout the paper we assume $\mathbb{R}$ be a set of real numbers and $\mathbb{R}_+$ a set of strictly positive real numbers. Let $E\subset\mathbb{R}_+^n$ represent the simplex

$$E=\{(x_1,x_2,\cdots,x_n):\sum\nolimits_{i=1}^n x_i\leqslant 1, x_i\geqslant 0, i=1,2,\cdots,n\},$$

and $x=(x_0,x_1,x_2,\cdots,x_n)$, where $x_0=1-\sum\nolimits_{i=1}^n x_i$. Let $dx=dx_1dx_2\cdots dx_n$ denote the differential of the volume in E.

Definition 2.1. *Let f be a positive real function and p a nonnegative integrable function on E, then the generalized weighted functional means of f with weight function p and two parameters r, s are defined by*

$$M_{r,s}(f;p)=\begin{cases}\left[\dfrac{\int_E p(x)f^r(x)dx}{\int_E p(x)f^s(x)dx}\right]^{\frac{1}{r-s}}, & r\neq s;\\ \exp\left\{\dfrac{\int_E p(x)f^r(x)\ln f(x)dx}{\int_E p(x)f^r(x)dx}\right\}, & r=s.\end{cases} \tag{2.1}$$

Lemma 2.1. (see [6]) *Suppose $f,g\geq 0$ and p are integrable, and f/g is continuous on E. Then there exists at least one point $v\in E$ such that*

$$\frac{\int_E p(x)f(x)dx}{\int_E p(x)g(x)dx}=\lim_{x\to v}\frac{f(x)}{g(x)}. \tag{2.2}$$

We call Lemma 2.1 the revised Cauchy's mean value theorem in the integral form.

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Theorem 2.1. *$M_{r,s}(f;p)$ has the following properties:*

$$\alpha \leqslant M_{r,s}(f;p) \leqslant \beta; \tag{2.3}$$

$$M_{r,s}(f;p) = M_{s,r}(f;p); \tag{2.4}$$

$$M_{r,s}^{r-s}(f;p) = M_{r,t}^{r-t}(f;p) \cdot M_{t,s}^{t-s}(f;p); \tag{2.5}$$

where $\alpha = \inf_{x\in E} f(x)$ and $\beta = \sup_{x\in E} f(x)$.

Proof. This follows from Definition 2.1, Lemma 2.1 and standard arguments. □

Lemma 2.2. (see [10, p. 26]) *Let $E \subset \mathbb{R}^n$, and $F(x), G(x) : E \to \mathbb{R}$ be integrable functions, satisfying $(F(u) - F(v))(G(u) - G(v)) \geqslant 0$ for any $u, v \in E$. Furthermore, if $g(x) : E \to [0, +\infty)$ is integrable, then*

$$\int_E g(x)F(x)dx \int_E g(x)G(x)dx \leqslant \int_E g(x)F(x)G(x)dx \int_E g(x)dx \tag{2.6}$$

with equality holding if and only if $F(x)$ and $G(x)$ are constants. If $(F(u) - F(v))(G(u) - G(v)) \leqslant 0$ for any $u, v \in E$, then the inequality (2.6) reverses.

Inequality (2.6) is called Tchebycheff's integral inequality.

Theorem 2.2. *$M_{r,s}(f;p)$ is increasing in both r and s.*

Proof. If $r = s$, then

$$M_{r,r}(f;p) = \exp\left\{\frac{\int_E p(x)f^r(x)\ln f(x)dx}{\int_E p(x)f^r(x)dx}\right\}.$$

By letting $F(x) = \ln f(x) > 0, G(x) = \ln f(x) > 0, g(x) = p(x)f^r(x) > 0$ in Lemma 2.2, we obtain

$$\frac{d}{dr}(\ln M_{r,r}(f;p)) = \frac{\int_E pf^r dx \int_E pf^r \ln^2 f dx - [\int_E pf^r \ln f dx]^2}{[\int_E pf^r dx]^2} > 0.$$

If $r \neq s$,

$$M_{r,s}(a) = \left[\frac{\int_E p(x)f^r(x)dx}{\int_E p(x)f^s(x)dx}\right]^{\frac{1}{r-s}},$$

we have

$$\frac{d}{dr}(\ln M_{r,s}(f;p)) = \frac{1}{(r-s)^2}\left[\frac{\int_E pf^r \ln f dx}{\int_E pf^r dx}\cdot(r-s) - \ln\left(\frac{\int_E pf^r dx}{\int_E pf^s dx}\right)\right].$$

Using the revised Lagrange's mean value theorem in the integral form, we get

$$\ln\left[\frac{\int_E p(x)f^r(x)dx}{\int_E p(x)f^s(x)dx}\right] = \frac{\int_E p(x)f^\zeta(x)\ln f(x)dx}{\int_E p(x)f^\zeta(x)dx}\cdot(r-s),$$

with ζ between p and q.

Let $F(x) = f^{r-\zeta}(x) > 0, G(x) = \ln f(x) > 0, g(x) = p(x)f^\zeta(x) > 0$ in Lemma 2.2, we obtain

$$\frac{\int_E p(x)f^\zeta(x)\ln f(x)dx}{\int_E p(x)f^\zeta(x)dx}\cdot(r-s) < \frac{\int_E p(x)f^r(x)\ln f(x)dx}{\int_E p(x)f^r(x)dx}\cdot(r-s),$$

that is $\frac{d}{dr}(\ln M_{r,s}(f;p)) > 0$, and $M_{r,s}(f;p)$ increases with both r and s since $M_{r,s}(f;p) = M_{s,r}(f;p)$. This completes the proof. □

Theorem 2.3. *Let f be a positive real function and p_1, p_2 nonnegative integral functions on E with $(p_1(u)/p_2(u) - p_1(v)/p_2(v))(f(u) - f(v)) \geqslant 0$ for any $u, v \in E$, then*

$$M_{r,s}(f;p_1) \geqslant M_{r,s}(f;p_2). \tag{2.7}$$

If $(p_1(u)/p_2(u) - p_1(v)/p_2(v))(f(u) - f(v)) \leqslant 0$ for any $u, v \in E$, inequality (2.7) reverses.

Proof. Inequality (2.7) follows by taking $g(x) = f^s(x)p_2(x), F(x) = p_1(x)/p_2(x)$ and $G(x) = f^{r-s}(x)$ in Lemma 2.2. □

3. The Three-Parameter Means in n Variables

In this section, let φ be a continuous function on an interval $\mathbb{I} \subseteq \mathbb{R}$, and $a_i \in \mathbb{I},\ a_i \neq a_j$ for $i \neq j$. Setting

$$V(a;\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}. \tag{3.1}$$

Let $\varphi(x) = x^{n+r} \ln^k x$ in (3.1), we have

$$V(a;r,k) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n+r} \ln^k a_0 \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n+r} \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^{n+r} \ln^k a_n \end{vmatrix}. \tag{3.2}$$

Note the case $r = 0$ and $k = 0$ is just the determinant of Van Der Monde's matrix of the n-th order:

$$V(a;0{,}0) = \sum\nolimits_{i=0}^{n} (-1)^{n+i} a_i^n V_i(a) = \prod_{0 \leqslant i < j \leqslant n} (a_j - a_i), \tag{3.3}$$

where

$$V_i(a) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{i-1} & a_{i-1}^2 & \cdots & a_{i-1}^{n-1} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix} \quad (0 \leqslant i \leqslant n). \tag{3.4}$$

Let $\ln a = (\ln a_0, \ln a_1, \cdots, \ln a_n)$, we denote

$$V(\ln a;r,k) = \begin{vmatrix} 1 & \ln a_0 & \ln^2 a_0 & \cdots & \ln^{n-1} a_0 & a_0^r \ln^k a_0 \\ 1 & \ln a_1 & \ln^2 a_1 & \cdots & \ln^{n-1} a_1 & a_1^r \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln a_n & \ln^2 a_n & \cdots & \ln^{n-1} a_n & a_n^r \ln^k a_n \end{vmatrix}. \tag{3.5}$$

Lemma 3.1. [16] *If $n \in \mathbb{N}$, a function φ be a n-th differentiable function on interval $\mathbb{I} \in \mathbb{R}_+$, then*

$$V(a;\varphi) = V(a;0,0) \int_E \varphi^{(n)} \left(\sum_{i=0}^{n} a_i x_i \right) dx. \tag{3.6}$$

Definition 3.1. *The three-parameter mean $E_{r,s,t}(a)$ is defined by*

$$E_{r,s,t}(a) = \begin{cases} \left[\dfrac{\int_E M_t^r(a;x)dx}{\int_E M_t^s(a;x)dx} \right]^{\frac{1}{r-s}}, & t(r-s) \neq 0; \\ \exp\left\{ \dfrac{\int_E M_t^r(a;x) \ln M_t(a;x)dx}{\int_E M_t^r(a;x)dx} \right\}, & t \neq 0, r = s; \\ \left[\dfrac{\int_E G^r(a;x)dx}{\int_E G^s(a;x)dx} \right]^{\frac{1}{r-s}}, & t = 0, r \neq s; \\ \exp\left\{ \dfrac{\int_E G^r(a;x) \ln G(a;x)dx}{\int_E G^r(a;x)dx} \right\}, & t = 0, r = s \neq 0; \\ \exp\left\{ n! \int_E \ln G(a;x)dx \right\}, & t = r = s = 0; \end{cases} \tag{3.7}$$

where $M_r(a;x) = \left[a_0^r + \sum_{i=1}^n (a_i^r - a_0^r)x_i\right]^{1/r} = \left(\sum_{i=0}^n a_i^r x_i\right)^{1/r}$, and $M_0(a;x) = G(a;x) = \prod_{i=0}^n a_i^{x_i}$.

Theorem 3.1. *Let $r, s, t \in \mathbb{R}$, then $E_{r,s,t}(a)$ is increasing with both r and s.*

Proof. This theorem follows from Definition 3.1, Theorem 2.2 and standard arguments. □

Theorem 3.2. *we have*

(3.8)
$$E_{r,s,t}(a)=\begin{cases}\left[\prod_{k=1}^{n}\left(\dfrac{kt+r}{kt+s}\right)\cdot\dfrac{V(a^t;r/t,0)}{V(a^t;s/t,0)}\right]^{\frac{1}{r-s}}, & t(r-s)\prod_{k=1}^{n}[(kt+r)(kt+s)]\neq 0;\\ \left[(-1)^{j+1}\dfrac{j}{n!}\cdot\dbinom{n}{j}\prod_{k=1}^{n}(kt+s)\cdot\dfrac{V(a^t;-j,1)}{V(a^t;s/t,0)}\right]^{\frac{-1}{jt+s}}, & \\ \qquad t(r-s)\prod_{k=1}^{n}(kt+r)\neq 0, r=-jt, 1\leqslant j\leqslant n; & \\ \left[(-1)^{j-k}\dfrac{j}{k}\dbinom{n}{j}\dbinom{n}{k}^{-1}\cdot\dfrac{V(a^t;-j,1)}{V(a^t;-k,1)}\right]^{\frac{1}{(k-j)t}}, & \\ \qquad t(r-s)\neq 0, r=-jt, s=-kt, 1\leqslant j\neq k\leqslant n; & \\ \exp\left\{\dfrac{V(a^t;r/t,1)}{V(a^t;r/t,0)}-\dfrac{1}{t}\sum_{j=1}^{n}\dfrac{1}{j+r/t}\right\}, & t\prod_{k=1}^{n}[(kt+r)(kt+s)]\neq 0, r=s;\\ \exp\left\{\dfrac{V(a^t;r,2)}{2V(a^t;r,1)}-\sum_{j=1,j\neq k}^{n}\dfrac{1}{j-k}\right\}, & t\neq 0, r=s=-kt, 1\leqslant k\leqslant n;\\ \left[\dfrac{s^n}{r^n}\cdot\dfrac{V(\ln a;r,0)}{V(\ln a;s,0)}\right]^{\frac{1}{r-s}}, & (r-s)rs\neq 0, t=0;\\ \left[\dfrac{n!}{r^n}\cdot\dfrac{V(\ln a;r,0)}{V(\ln a;0,n)}\right]^{1/r}, & r\neq 0, t=s=0;\\ \exp\left\{\dfrac{V(\ln a;r,1)}{V(\ln a;r,0)}-\dfrac{n}{r}\right\}, & r=s\neq 0, t=0;\\ \left(\prod_{i=0}^{n}a_i\right)^{1/(n+1)}, & t=r=s=0;\end{cases}$$

where $a=(a_0,a_1,\cdots,a_n)\in\mathbb{R}_+^{n+1}$, $a^t=(a_0^t,a_1^t,\cdots,a_n^t)$, $\ln a=(\ln a_0,\ln a_1,\cdots,\ln a_n)$ *and* $a_i\neq a_j$ *for* $1\leqslant i\neq j\leqslant n$*.*

Proof. For $t\neq 0$, setting two functions

$$\varphi_1(r,t;x)=\begin{cases}\prod_{k=1}^{n}(k+r/t)^{-1}x^{n+r/t}, & r+kt\neq 0, 1\leqslant k\leqslant n;\\ \left[(-1)^{k+1}(k-1)!(n-k)!\right]^{-1}x^{n-k}\ln x, & r+kt=0, 1\leqslant k\leqslant n,\end{cases}$$

$$\varphi_2(r,t;x)=\begin{cases}\prod_{k=1}^{n}(k+r/t)^{-1}x^{n+r/t}\left(\ln x-\sum_{j=1,j\neq k}^{n}\dfrac{1}{j+r/t}\right), & r+kt\neq 0, 1\leqslant k\leqslant n;\\ \left[(-1)^{k+1}(k-1)!(n-k)!\right]^{-1}x^{n-k}\ln x\left(\dfrac{\ln x}{2}-\sum_{j=1,j\neq k}^{n}\dfrac{1}{j-k}\right), & r+kt=0, 1\leqslant k\leqslant n;\end{cases}$$

then $\varphi_1^{(n)}(r,t;x)=x^{r/t}$, and $\varphi_2^{(n)}(r,t;x)=x^{r/t}\ln x$.

Let $f^r(x)=\varphi_1^{(n)}(r,t;x)$, $f^s(x)=\varphi_1^{(n)}(s,t;x)$, and $f^r(x)\ln f(x)=\varphi_2^{(n)}(r,t;x)$. Then this theorem follows from Definition 3.1, Lemma 3.1 and straightforward computation.

For $t=0$, taking $\varphi_1(r;x)=r^{-n}e^{rx}$, $\varphi_2(r;x)=r^{-n}xe^{rx}-nr^{-n-1}e^{rx}$, then $\varphi_1^{(n)}(r;x)=e^{rx}$, $\varphi_2^{(n)}(r;x)=xe^{rx}$, which similarly imply Theorem 3.2. □

Remark 3.1. *$E_{r,s,t}(a)$ include that Alzer mean $J_r(a)$, Stolarsky mean $S_r(a)$, Pečarić-Šimić mean $S_{r,t}(a)$ and two-parameter mean $E_{r,s}(a)$ in $n+1$ variables $a_0,a_1,\cdots,a_n$ are defined in* [12]-[17]*, i.e.*

(3.9)
$$J_r(a)=E_{r,r-1,1}(a)=E_{r,r-1,1}(a)=\frac{\int_E A^r(a;x)dx}{\int_E A^{r-1}(a;x)dx},$$

$$S_r(a) = E_{r,0,1}(a) = \begin{cases} \left[n! \cdot \int_E A^r(a;x)dx\right]^{\frac{1}{r}}, & r \neq 0; \\ \exp\left\{n! \cdot \int_E A^r(a;x) \ln A(a;x)dx\right\}, & r = 0; \end{cases} \tag{3.10}$$

$$J_{s,t}(x) = E_{r,r-1,t}(a) = \begin{cases} \dfrac{\int_E M_t^r(a;x)dx}{\int_{E}, M_t^{r-1}(a;x)dx}, & t \neq 0; \\ \dfrac{\int_E G^r(a;x)dx}{\int_E G^{r-1}(a;x)dx}, & t = 0. \end{cases} \tag{3.11}$$

$$S_{r,t}(a) = E_{r,0,t}(a) = \begin{cases} \left[n! \cdot \int_E M_t^r(a;x)dx\right]^{\frac{1}{r}}, & tr \neq 0; \\ \exp\left\{n! \cdot \int_E \ln M_t(a;x)dx\right\}, & t \neq 0, r = 0; \\ \left[\int_E G^r(a;x)dx\right]^{\frac{1}{r}}, & t = 0, r \neq 0; \\ \exp\left\{n! \cdot \int_E \ln G(a;x)dx\right\}, & r = t = 0; \end{cases} \tag{3.12}$$

and

$$E_{r,s}(a) = E_{r,s,1}(a) = \begin{cases} \left[\dfrac{\int_E A^r(a;x)dx}{\int_E A^s(a;x)dx}\right]^{\frac{1}{r-s}}, & r \neq s; \\ \exp\left\{\dfrac{\int_E A^r(a;x) \ln A(a;x)dx}{\int_E A^r(a;x)dx}\right\}, & r = s; \end{cases} \tag{3.13}$$

respectively, where the notions of $E, M_t(a;x), G(a;x)$ are the same as in the Definition 3.1. We have also

$$J_r(a) = \begin{cases} \dfrac{r}{n+r} \cdot \dfrac{V(a,r;0)}{V(a,r-1;0)}, & r \neq 0, -1, \cdots, -n; \\ \dfrac{V(a,0;0)}{nV(a,-1;1)}, & r = 0; \\ \dfrac{r}{n+r} \cdot \dfrac{V(a,r;1)}{V(a,r-1;1)}, & r = -1, \cdots, -(n-1); \\ \dfrac{-nV(a,0;1)}{V(a,-n-1;0)}, & r = -n; \end{cases} \tag{3.14}$$

$$S_r(a) = \begin{cases} \left[\dfrac{n!}{\prod_{k=1}^n (k+r)} \cdot \dfrac{V(a;r,0)}{V(a;0,0)}\right]^{\frac{1}{r}}, & r \neq 0, -1, -2, \cdots, -n; \\ \exp\left\{\dfrac{V(a;0,1)}{V(a;0,0)} - \sum_{k=1}^n \dfrac{1}{k}\right\}, & r = 0; \\ \left[\dfrac{n! \cdot V(a;r,1)}{(-1)^{r+1}(-r-1)! \cdot (n+r)! \cdot V(a;0,0)}\right]^{\frac{1}{r}}, & r = -1, \cdots, -n; \end{cases} \tag{3.15}$$

$$J_{r,t}(a) = \begin{cases} \prod_{k=1}^n \left(\dfrac{kt+r-1}{kt+r}\right) \cdot \dfrac{V(a^t;rt,0)}{V(a^t;(r-1)/t,0)}, & t\prod_{k=1}^n [(kt+r)(kt+r-1)] \neq 0; \\ (-1)^{j+1} j \cdot \dfrac{1}{n!} \cdot \binom{n}{j} \prod_{k=1}^n (kt+r-1) \cdot \dfrac{V(a^t;-j,1)}{V(a^t;(r-1)/t,0)}, & \\ & t\prod_{k=1}^n (kt+r-1) \neq 0, r = -jt, 1 \leqslant j \leqslant n; \\ (-1)^{j-k} \dfrac{j}{k} \binom{n}{j} \binom{n}{k}^{-1} \cdot \dfrac{V(a^t;-j,1)}{V(a^t;-k,1)}, & t \neq 0, r = -jt, (k-j)t = 1, 1 \leqslant j,k \leqslant n; \\ \left(\dfrac{r-1}{r}\right)^n \cdot \dfrac{V(\ln a;r,0)}{V(\ln a;r-1,0)}, & r(r-1) \neq 0, t = 0; \\ \dfrac{n!V(\ln a;1,0)}{V(\ln a;0,n)}, & r = 1, t = 0; \\ \dfrac{(-1)^n V(\ln a;0,n)}{n!V(\ln a;-1,0)}, & r = 0, t = 0; \end{cases} \tag{3.16}$$

$$
(3.17) \qquad S_{r,t}(a) = \begin{cases} \left[\dfrac{n! \cdot t^n}{\prod_{k=1}^{n}(kt+r)} \cdot \dfrac{V(a^t; r/t, 0)}{V(a^t; 0, 0)}\right]^{\frac{1}{r}}, & t \neq 0, r \neq -kt, 0 \leqslant k \leqslant n, k \in \mathbb{N}; \\ \left[(-1)^{k+1} kt \binom{n}{k} \dfrac{V(a^t; -k, 1)}{V(a^t; 0, 0)}\right]^{\frac{1}{-kt}} & t \neq 0, r = -kt, 1 \leqslant k \leqslant n, k \in \mathbb{N}; \\ \left[\dfrac{n!}{r^n} \cdot \dfrac{V(\ln a; r, 0)}{V(\ln a; 0, n)}\right]^{\frac{1}{r}}, & t = 0, r \neq 0; \\ \exp\left\{\dfrac{V(a^t; 0, 1)}{V(a^t; 0, 0)} - \dfrac{1}{t}\displaystyle\sum_{k=1}^{n}\dfrac{1}{k}\right\}, & r = 0, t \neq 0; \\ \left(\prod_{i=0}^{n} a_i\right)^{1/(n+1)}, & r = t = 0; \end{cases}
$$

and

$$
(3.18) \qquad E_{r,s}(a) = \begin{cases} \left[\prod_{k=1}^{n}\left(\dfrac{k+r}{k+s}\right) \cdot \dfrac{V(a; r, 0)}{V(a; s, 0)}\right]^{\frac{1}{r-s}}, & (r-s)\prod_{k=1}^{n}[(k+r)(k+s)] \neq 0; \\ \left[\dfrac{(-1)^{s+1}(-s-1)!(s+n)!}{\prod_{k=1}^{n}(k+r)} \cdot \dfrac{V(a; r, 0)}{V(a; s, 1)}\right]^{\frac{1}{r-s}}, & r \neq s = -1, -2, \cdots, -n; \\ \left[\dfrac{(-1)^{r-s}(-s-1)!(s+n)!}{(-r-1)!(r+n)!} \cdot \dfrac{V(a; r, 1)}{V(a; s, 1)}\right]^{\frac{1}{r-s}}, & r \neq s; r, s = -1, -2, \cdots, -n; \\ \exp\left\{\dfrac{V(a; r, 1)}{V(a; r, 0)} - \displaystyle\sum_{k=1}^{n}\dfrac{1}{k+r}\right\}, & r = s \neq -1, -2, \cdots, -n; \\ \exp\left\{\dfrac{V(a; r, 2)}{2V(a; r, 1)} - \displaystyle\sum_{k=1, k\neq -r}^{n}\dfrac{1}{k+r}\right\}, & r = s = -1, -2, \cdots, -n; \end{cases}
$$

where $a = (a_0, a_1, \cdots, a_n) \in \mathbb{R}_+^{n+1}$, $a^t = (a_0^t, a_1^t, \cdots, a_n^t)$, $\ln a = (\ln a_0, \ln a_1, \cdots, \ln a_n)$ *and* $a_i \neq a_j$ *for* $1 \leqslant i \neq j \leqslant n$.

Theorem 3.3. *If* $E_{r,s,t}(a)$ *be the three-parameter mean, then*

(a) $E_{r,s,t}(a) = E_{s,r,t}(a)$;

(b) $\lim_{r\to\infty} E_{r,s,t}(a) = \lim_{s\to\infty} E_{s,r,t}(a) = \lim_{t\to\infty} E_{r,s,t}(a) = a_{\max}$;

(c) $\lim_{r\to-\infty} E_{r,s,t}(a) = \lim_{s\to-\infty} E_{s,r,t}(a) = \lim_{t\to-\infty} E_{r,s,t}(a) = a_{\min}$;

(d) $\lim_{r\to s} E_{r,s,t}(a) = E_{r,r,t}(a)$;

(e) $a_{\min} \leqslant E_{r,s,t}(a) \leqslant a_{\max}$;

(f) $E_{r,s,t}(a) = a_0$ *if and only if* $a_0 = a_1 = \cdots = a_n$;

(g) $E_{r,s,t}(ta) = tE_{r,s,t}(a), t > 0,$ *when* $ta = (ta_0, ta_1, \cdots, ta_n)$;

(h) $(E_{r,s,t}(a))^{r-s} = (E_{r,u,t}(a))^{r-u} \cdot (E_{u,s,t}(a))^{u-s}$.

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