ON THE INTERSECTION OF SOME FAMILIES OF MEANS

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Abstract. Alzer and Ruscheweyh have shown that any mean that is simultaneously a Gini and a Stolarsky mean belongs to the family of power means. We extend this result by proving analogous result for a broad class of two-parameter means, called S-means.

1. Introduction

For real r, s and positive x, y the Gini means are defined by

$$G(r,s;x,y) = \left(\frac{A(x^s,y^s)}{A(x^r,y^r)}\right)^{1/(s-r)} = \left(\frac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)} \tag{1.1}$$

where $A(x,y)$ is the arithmetic mean.
Exactly in the same way K.B. Stolarsky [6] defined extended mean values taking the logarithmic mean as the base:

$$E(r,s;x,y) = \left(\frac{L(x^s,y^s)}{L(x^r,y^r)}\right)^{1/(s-r)} = \left(\frac{r}{s}\frac{x^s-y^s}{x^r-y^r}\right)^{1/(s-r)} \tag{1.2}$$

The Gini means can be easily defined for $\mathbf{x} \in \mathbf{R}_+^n$:

$$G(r,s;\mathbf{x}) = \left(\frac{A(x_1^s,\dots,x_n^s)}{A(x_1^r,\dots,x_n^r)}\right)^{1/(s-r)} = \left(\frac{x_1^s+\dots+x_n^s}{x_1^r+\dots+x_n^r}\right)^{1/(s-r)} \tag{1.3}$$

Unfortunately, the there is no such obvious generalisation of the logarithmic mean in n variables (for some possible definitions see [5, 4]). Stolarsky [6] defined E as follows:

$$E(r,s;\mathbf{x}) = \left(\frac{(r-n+2)_{n-1}\sum_{k=1}^n x_k^s/Q'(x_k)}{(s-n+2)_{n-1}\sum_{k=1}^n x_k^s/Q'(x_k)}\right)^{1/(s-r)} \tag{1.4}$$

where $Q(x) = (x-x_1)\dots(x-x_n)$ and $(a)_b = \Gamma(a+b)/\Gamma(a)$.

Three classical means: arithmetic, geometric and harmonic, belong to both families:

$$A(\mathbf{x}) = G(0,1;\mathbf{x}) = E(n-1,n;\mathbf{x}),$$
$$G(\mathbf{x}) = G(0,0;\mathbf{x}) = E(-1,n-1;\mathbf{x}),$$
$$H(\mathbf{x}) = G(-1,0;\mathbf{x}) = E(-1,-2;\mathbf{x}),$$

D.H. Lehmer in [3] used the MacLaurin expansion of $E(r,2r;1+t,1)$ and $G(u,u-1;1+t,1)$ to show the above means are the only common elements of the two families. H.W. Gould and M.E. Mays ([2]) obtained the same result for $E(r,s;x,y)$ and $G(u,u-1;x,y)$.

H. Alzer and S. Ruscheweyh applied the hypergeometric functions to solve the problem completely:

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Theorem 1.1 ([1]). *The only means which belong to both $G(u,v;\mathbf{x})$ and $E(r,s;\mathbf{x})$ are*

(i) *the power means, if $n=2$, and*
(ii) *the arithmetic, geometric and harmonic means, if $n\geq 3$.*

In case $n=2$ the Stolarsky means and the Gini means are members of the family of S-means defined as

$$S(\alpha;r,s;x,y)=\frac{E(r,s;x^{\alpha+1};y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})}. \tag{1.5}$$

One can easily verify that $E(r,s)=S(0;r,s)$ and $G(r,s)=S(1;r,s)$. S-means contain some other interesting two-parameter means:
Heronian

$$S(1/2;r,s;x,y)=\left(\frac{x^s+\sqrt{xy}^s+y^s}{x^r+\sqrt{xy}^r+y^r}\right)^{1/(s-r)},$$

and generalized Heronian

$$S(1/n;r,s;x,y)=\left(\frac{\sum_{i=0}^n x^{\frac{n-i}{n}s}y^{\frac{i}{n}s}}{\sum_{i=0}^n x^{\frac{n-i}{n}r}y^{\frac{i}{n}r}}\right)^{1/(s-r)}$$

The goal of this note is to investigate the intersection of the families $E(r,s;x,y)$ and $S(\alpha;u,v;x,y)$.

2. Lemmas

We start with some basic properties of E and S.

Lemma 2.1. *S is a mean, i.e. for every real α and $x<y$*

$$x\leq S(\alpha;r,s;x,y)\leq y.$$

Proof. $E(r,s;x,y)$ is increasing in all variables and homogeneous of degree 1 in x and y, so for fixed positive a,b we have

$$x=\frac{E(r,s;ax,bx)}{E(r,s;a,b)}\leq\frac{E(r,s;ax,by)}{E(r,s;a,b)}\leq\frac{E(r,s;ay,by)}{E(r,s;a,b)}=y.$$

Setting $a=x^\alpha, b=y^\alpha$ completes the proof. □

The property below follows immediately from the homogeneity of E;

Property 2.2.

$$E(-r,-s;x,y)=\frac{xy}{E(r,s;x,y)}, \tag{2.1}$$

$$S(\alpha;-r,-s;x,y)=\frac{xy}{S(\alpha;r,s;x,y)}. \tag{2.2}$$

Lemma 2.3. *For $r+s\neq 0$*

$$E(-r,s;x,y)=(xy)^{\frac{r}{s+r}}E^{\frac{s-r}{s+r}}(r,s;x,y) \tag{2.3}$$

$$S(\alpha;-r,s;x,y)=(xy)^{\frac{r}{s+r}}S^{\frac{s-r}{s+r}}(\alpha;r,s;x,y) \tag{2.4}$$

Proof.

$$E(-r,s;x,y)=\left(\frac{-r}{s}\frac{x^s-y^s}{x^{-r}-y^{-r}}\right)^{1/(s+r)}=(xy)^{\frac{r}{s+r}}\left(\frac{r}{s}\frac{x^s-y^s}{x^r-y^r}\right)^{1/(s+r)}$$
$$=(xy)^{\frac{r}{s+r}}E^{\frac{s-r}{s+r}}(r,s;x,y).$$

(2.4) follows immediately from (2.3) and the definition of S. □

By a straightforward computation we see that for $r,s>0$

(2.5) $$E(r,s;0,1)=\lim_{x\to 0}E(r,s;x,1)=\begin{cases}(r/s)^{1/(s-r)} & s\neq r\\ \exp(-1/s) & r=s\end{cases}$$

and if $\alpha>0$

(2.6) $$S(\alpha;r,s;0,1)=\lim_{x\to 0}S(\alpha;r,s;x,1)=1.$$

The fact that for $r,s>0$ $E(r,s;0,1)\neq 1$ will play the crucial role in the proof of the main theorem.

We write $f\approx g$ if $\lim_{x\to 0^+}f(x)/g(x)=1$.

Two lemmas that follow describe behaviour of E and S for small values of x.

Lemma 2.4. *If $r+s\neq 0$ then there exist constants a,b, and $C>0$ such that either*

$$E(r,s;x,1)\approx Cx^a|\log x|^b \qquad \text{with } b\neq 0$$

or

$$E(r,s;x,1)\approx Cx^a \qquad \text{with } C\neq 1$$

Proof. Consider five cases

$r,s>0$: $a=b=0, C=E(r,s;0,1)$ by (2.5)
$r,s<0$: $a=1,b=0,C=1/E(r,s;0,1)$ by (2.1)
$r<0<s$: $a=\frac{r}{s+r},b=0,C=E^{\frac{s-r}{s+r}}(r,s;0,1)$. by (2.3)
$0=r<s$: $E(0,s;x,1)=\left(\frac{x^s-1}{s\log x}\right)^{1/s}$, hence $a=0,b=-1/s,C=s^{-1/s}$
$r<s=0$: $E(0,r;x,1)=\left(\frac{x^r-1}{r\log x}\right)^{1/r}=\left(x^r\frac{1-x^{-r}}{r\log x}\right)^{1/r}$, hence $a=1,b=-1/r$, $C=(-r)^{-1/r}$

□

Lemma 2.5. *If $rs(r+s)\neq 0$ and $\alpha>0$, then there exists a constant d such that*

$$S(\alpha;r,s;x,1)\approx x^d.$$

Proof.

$$d=\begin{cases}0 & \text{if } r,s>0 \text{ by (2.6)}\\ 1 & \text{if } r,s<0 \text{ by (2.2)}\\ \frac{r}{s+r} & \text{if } r<0<s \text{ by (2.4)}\end{cases}.$$

□

3. Main result

We have already all the tools to prove the main result of this note

Theorem 3.1. *If $\alpha > 0$ then the intersection of the families of means*

$$S_0 = \{E(r, s; x, y) : r, s \in \mathbf{R}\}$$

and

$$S_\alpha = \{S(\alpha, u, v; x, y) : u, v \in \mathbf{R}\}$$

consists of the family of means

$$T_\alpha = \{E((\alpha + 1)t, \alpha t; x, y) : t \in \mathbf{R}\} \tag{3.1}$$

Proof. Suppose that $E(r, s; x, y) \equiv S(\alpha, u, v; x, y)$. If $r + s = 0$ then the both sides are equal to the geometric mean, which is $E(0, 0; x, y)$ and clearly belongs to T_α. Suppose now that $r + s \neq 0$. If $(u + v)uv \neq 0$ then by Lemmas 2.4 and 2.5 we would have $Cx^a|\log x|^b \approx x^d$ or $Cx^a \approx x^d$, but both are impossible (in the second case $C \neq 1$. Hence $u + v = 0$ which leads again to the geometric mean, or $uv = 0$. In this case we have

$$\begin{aligned} S(\alpha, 0, v; x, y) &= \frac{E(0, v; x^{\alpha+1}, y^{\alpha+1})}{E(0, v; x^\alpha, y^\alpha)} \\ &= \left(\frac{x^{(\alpha+1)v} - y^{(\alpha+1)v}}{v(\log x^{\alpha+1} - \log y^{\alpha+1})}\right)^{1/v} \Big/ \left(\frac{x^{\alpha v} - y^{\alpha v}}{v(\log x^\alpha - \log y^\alpha)}\right)^{1/v} \\ &= \left(\frac{\alpha v}{(\alpha+1)v}\frac{x^{(\alpha+1)v} - y^{(\alpha+1)v}}{x^{\alpha v} - y^{\alpha v}}\right)^{1/((\alpha+1)v - \alpha v)} = E(\alpha v, (\alpha + 1)v; x, y) \end{aligned}$$

hence both means belong to T_α. This equation also shows, that every mean from T_α belongs to both S_0 and S_α. The proof is thus complete. □

The following identity holds

$$S(\alpha; r, s; x, y) = (xy)^{-\alpha} S^{2\alpha+1}(\tfrac{-\alpha}{2\alpha+1}; (2\alpha + 1)r, (2\alpha + 1)s; x, y)$$

which shows that the Lemma 2.5 is valid for $-1/2 < \alpha < 0$. Moreover, because

$$S(-\tfrac{1}{2} - \alpha)S(-\tfrac{1}{2} + \alpha) = S^2(-\tfrac{1}{2}) = xy$$

the Lemma 2.5 remains valid for $\alpha \in (-\infty, -1/2)\backslash\{-1\}$, which implies that the Theorem 3.1 holds for all $\alpha \in \mathbf{R}\backslash\{-1, -1/2, 0\}$.
To complete the analysis observe that $S_{-1} = S_0$ and $S_{-1/2} = \{\sqrt{xy}\}$.

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