ON THE COMPOSITION OF COMPLETELY MONOTONIC FUNCTIONS

SENLIN GUO

Abstract. In this article, we investigate the composition of functions related to the completely monotonic functions.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := (0, \infty)$, I^+ is an open interval contained in $\mathbb{R}^+$, I^o is the interior of the interval $I \subset \mathbb{R}$, $-I := \{-x |\ x \in I\}$, $\mathcal{R}(f)$ denotes the range of the function f and $C(I)$ is the class of all continuous functions on I.

We first recall some definitions.

Definition 1.1 ([5]). *A function f is said to be absolutely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

We use $AM(I)$ to denote the class of all absolutely monotonic functions on I.

Definition 1.2 ([5]). *A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Some mathematicians use the terminology of completely monotone instead of completely monotonic.

The class of all completely monotonic functions on I is denoted by $CM(I)$.

Definition 1.3 ([4]). *A function f is said to be strongly completely monotonic on I^+ if, for all $n \in \mathbb{N}_0, (-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on I^+.*

The class of such functions is denoted by $SCM(I^+)$.

Definition 1.4 ([1]). *A function f is said to be almost completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Let's use $ACM(I)$ to denote the set of all such functions on I.

Definition 1.5 ([1]). *A function f is said to be almost strongly completely monotonic on $\mathbb{R}^+$ if, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $\mathbb{R}^+$.*

2000 *Mathematics Subject Classification.* Primary 26A48, 26E40; Secondary 33B15.

Key words and phrases. Composition, completely monotonic function, strongly completely monotonic function.

The class of almost strongly completely monotonic functions on $\mathbb{R}^+$ is denoted by $ASCM(\mathbb{R}^+)$.

For compositions of completely monotonic and related functions. The following two results are a version of the corresponding Theorems in [5, Chapter IV]

Theorem 1.6. *Suppose that* $f \in AM(I_1), g \in AM(I)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in AM(I)$.

Theorem 1.7. *Suppose that* $f \in AM(I_1), g \in CM(I)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in CM(I)$.

The next result, which was established in 1983 by Lorch and Newman [2, Theorem 5], is a converse of Theorem 1.7.

Theorem 1.8. *Let* f *be defined on* $[0, \infty)$. *If for each* $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$*, then* $f \in AM(\mathbb{R}^+)$.

The following result is a generalized form of Theorem 2 of [3].

Theorem 1.9. *Suppose that* $f \in CM(I_1), g \in C(I), g' \in CM(I^0)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in CM(I)$.

From this result we can obtain

Corollary 1.10. *If* $f \in CM(I_1)$*,where* $I_1 := (a, b)$ *with* $-\infty \leq a < b < \infty$*, then* $f(b - e^{-x}) \in CM(I)$. *Here* $I := (-\ln(b - a), \infty)$.

In 1983 Lorch and Newman [2, Theorem 4] gave an interesting result related to Theorem 1.9 as follows.

Theorem 1.11. *For each function* $f \in CM(I)$*, where* $I := [0, \infty)$*, there exists a function* g *on* I *such that* $g(0) = 0, f \circ g \in CM(I)$ *and* $g' \notin CM(\mathbb{R}^+)$.

2. Main results

In this article, we further investigate the composition of functions related to the completely monotonic functions.

First we give the following

Definition 2.1. *A function* f *is said to be almost absolutely monotonic on an interval* I*, if* $f \in C(I)$*, has derivatives of all orders on* I^o *and for all* $n \in \mathbb{N}$

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Let's use $AAM(I)$ to denote the class of all such functions on I.

From definitions we directly have

Theorem 2.2. *A function* f *is almost completely monotonic on an interval* I *if and only if the function* $f(-x)$ *is almost absolutely monotonic on* $-I$.

We can prove the following results of the compositions.

Theorem 2.3. *Suppose that* $f \in ACM(I_1), g \in C(I), g' \in CM(I^0)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in ACM(I)$.

Theorem 2.4. *Suppose that* $f \in ASCM(I_1^+), g' \in SCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1^+$. *If* $2xg'(x) \geq g(x), x \in I^+$*, then* $f \circ g \in ASCM(I^+)$.

Remark 2.5. *The condition* $2xg'(x) \geq g(x), x \in I^+$ *in Theorem 2.3(2) can not be deleted even if* $f \in ASCM(I_1^+)$ *is replaced by a stronger condition:* $f \in SCM(I_1^+)$*. For example, let* $f(x) := 1/x, g(x) := \ln x$ *and* $I^+ := (e^2, \infty)$*. Then* $f \in SCM(\mathbb{R}^+), g' \in SCM(I^+)$*, and the condition* $2xg'(x) \geq g(x), x \in I^+$ *is not satisfied. It is easy to show that*

$$(-1)^1 x^2 [f \circ g(x)]' = \frac{x}{\ln^2 x}$$

is strictly increasing on I^+*. Hence* $f \circ g(x) \notin ASCM(I^+)$*.*

Theorem 2.6. *Suppose that* $f \in ACM(I_1), -g \in ACM(I)$ *and* $\mathcal{R}(g) \subset I_1$*, then* $f \circ g \in ACM(I)$*.*

Theorem 2.7. *Suppose that* $f \in ACM(I_1), -g \in ASCM(I^+)$ *and* $\mathcal{R}(g) \subset I_1$*. Then* $f \circ g \in ASCM(I^+)$*.*

References

[1] S. Guo, Some function classes connected to the class of completely monotonic functions, *RGMIA Res. Rep. Coll.* **9**(2) (2006), Art. 8 [Online: http://rgmia.vu.edu.au/v9n2.html].

[2] L. Lorch, and D.J. Newman, On the composition of completely monotonic functions completely monotonic sequences and related questions, *J. London Math. Soc.(2)* **28** (1983), 31–45.

[3] K. S. Miller and S. G. Samko, Completely monotonic functions , *Integral Transform. Spec. Funct.* **12** (2001), 389–402.

[4] S.Y. Trimble, J. Wells and F.T. Wright, Supperaddative functions and a statistical application, *SIAM J. Math. Anal.* **20** (1989), 1255–1259.

[5] D.V. Widder, *The Laplace transform*, 7th printing, Pinceton University Press, Pinceton, 1966.

Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada

E-mail address: sguo@hotmail.com; umguos@cc.umanitoba.ca