SOME FUNCTION CLASSES CONNECTED TO THE CLASS OF COMPLETELY MONOTONIC FUNCTIONS

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Abstract. In this article, we introduce some function classes connected to the class of completely monotonic functions.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := (0, \infty)$ and I^o is the interior of the interval I.

We first recall some definitions.

Definition 1 ([17]). *A function f is said to be* absolutely monotonic *on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

We use $AM(I)$ to denote the class of all *absolutely monotonic* functions on I.

Definition 2 ([17]). *A function f is said to be* completely monotonic *on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Some mathematicians use the terminology of *completely monotone* instead of *completely monotonic*.

The class of all *completely monotonic* functions on I is denoted by $CM(I)$. From Definitions 1 and 2, we have

Theorem 1 ([17]). *A function f is* completely monotonic *on an interval I, if and only if the function $f(-x)$ is* absolutely monotonic *on $-I$.*

Definition 3 ([12]). *A function f is said to be* logarithmically completely monotonic *on an interval I if $f > 0, f \in C(I)$, has derivatives of all orders on I^o and for $n \in \mathbb{N}$*

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, \quad x \in I^o.$$

The set of all *logarithmically completely monotonic* functions on I is denoted by $LCM(I)$.

It was proved [12] that

Theorem 2.

$$LCM(I) \subset CM(I).$$

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The following example shows that

$$LCM(I) \neq CM(I).$$

Let $f(x) := -\ln x, I := (0, 1)$. Then $f \in CM(I)$. But $f \notin LCM(I)$ since

$$(\ln f(x))'' = \frac{-(1 + \ln x)}{(x \ln x)^2} < 0$$

if $x \in (1/e, 1)$.

The following form of definition of Stieltjes transform was adopted in [6, p. 127].

Definition 4. *A function f on $\mathbb{R}^+$ is called a Stieltjes transform, if there exist a constant $a \geq 0$ and a positive measure μ on $[0, \infty)$ such that*

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x + t}, \quad x \in \mathbb{R}^+.$$

For example, let $f(x) := 1/\sqrt{x}$. Since

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{dt}{(x + t)\sqrt{t}}, \quad x \in \mathbb{R}^+,$$

$f(x)$ is a Stieltjes transform.

We note that this definition of Stieltjes transform is a little different from that in [18], [17, p.325]. In the latter, a Stieltjes transform may be negative.

It was proved (see [5, Theorem 1.2])

Theorem 3. *A non-zero Stieltjes transform belong to $LCM(\mathbb{R}^+)$.*

Definition 5 ([14])**.** *A function f is said to be strongly completely monotonic on $\mathbb{R}^+$ if, for all $n \in \mathbb{N}_0, (-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $\mathbb{R}^+$.*

The class of such functions is denoted by $SCM(\mathbb{R}^+)$. Clearly,

$$SCM(\mathbb{R}^+) \subset CM(\mathbb{R}^+).$$

Let $\alpha \geq 1$, then the functions $f_\alpha(x) := x^{-\alpha} \in LCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+)$. Therefore

$$LCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) \neq \emptyset.$$

2. Some function classes

Now we introduce some function classes connected to the class of completely monotonic functions.

Definition 6. *A function f is said to be* strongly logarithmically completely monotonic *on $\mathbb{R}^+$ if $f > 0$ and, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1} [\ln f(x)]^{(n)}$ are nonnegative and decreasing on $\mathbb{R}^+$.*

Such a function class is denoted by $SLCM(\mathbb{R}^+)$. For example, $e^{1/x} \in SLCM(\mathbb{R}^+)$. From the definition,

$$SLCM(\mathbb{R}^+) \subset LCM(\mathbb{R}^+).$$

Definition 7. *A function f is said to be* almost strongly completely monotonic *on $\mathbb{R}^+$ if, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1} f^{(n)}(x)$ are nonnegative and decreasing on $\mathbb{R}^+$.*

The class of almost strongly completely monotonic functions on $\mathbb{R}^+$ is denoted by $ASCM(\mathbb{R}^+)$. Clearly $SCM(\mathbb{R}^+) \subset ASCM(\mathbb{R}^+)$.

Definition 8. *A function f is said to be* almost completely monotonic *on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}$*

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Let's use $ACM(I)$ to denote the set of all such functions on I.

It is easy to see that

$$ASCM(I^+) \cup CM(I^+) \subset ACM(I^+). \tag{2.1}$$

The following two results immediately follow from definitions.

Theorem 4.

(1) $-f \in ACM(I)$ *if and only if* $f \in C(I)$ *and* $f' \in CM(I^o)$.

(2) $-f \in ASCM(\mathbb{R}^+)$ *implies* $f' \in SCM(\mathbb{R}^+)$.

We note that the converse of (2) Theorem 4 is not true. For example, let $f(x) := \ln x$, then $f'(x) = 1/x \in SCM(\mathbb{R}^+)$, while $-f(x) = -\ln x \notin ASCM(\mathbb{R}^+)$ – indeed, $(-1)^1 x^2[-f(x)]' = x$ is strictly increasing on $\mathbb{R}^+$.

Theorem 5.

(1) $f \in LCM(I)$ *if and only if* $f > 0$ *and* $\ln f \in ACM(I)$.

(2) $f \in SLCM(\mathbb{R}^+)$ *if and only if* $f > 0$ *and* $\ln f \in ASCM(\mathbb{R}^+)$.

The following result can be proved by induction.

Theorem 6.

$$SLCM(\mathbb{R}^+) \subset ASCM(\mathbb{R}^+). \tag{2.2}$$

In words, a strongly logarithmically completely monotonic function on $\mathbb{R}^+$ must be almost strongly completely monotonic on $\mathbb{R}^+$.

Standard argument can show

Theorem 7.

$$SLCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) = \emptyset.$$

In words, a strongly logarithmically completely monotonic function on $\mathbb{R}^+$ must not be strongly completely monotonic on $\mathbb{R}^+$, or, a strongly completely monotonic function on $\mathbb{R}^+$ must not be strongly logarithmically completely monotonic on $\mathbb{R}^+$.

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