A NEW HILBERT'S INEQUALITY WITH LOGARITHM

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ABSTRACT. In this paper we introduce a new type of the Hardy-Hilbert's inequality with logarithm. This allows us to deduce some other new inequalities as applications.

1. INTRODUCTION

Suppose (a_n) and (b_n) are two arbitrary sequences of real numbers, if $0 < \sum_{n=0}^{\infty} a_n^2 < \infty, 0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\{\sum_{n=0}^{\infty} a_n^2\}^{1/2}\{\sum_{n=0}^{\infty} b_n^2\}^{1/2} \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is call the slightly sharper form of the well known Hardy-Hilbert's inequality(see [1]). The integral analogue can be stated as follows

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y+1}dxdy < \pi\{\int_0^{\infty} f^2(x)dx\int_0^{\infty} g^2(x)dx\}^{\frac{1}{2}} \tag{1.2}$$

where the constant factor π is the best possible. Hardy-Hilbert's inequality is important in analysis, many refinements and applications have been given later on (see [2], [5], [7]).

In 1997, Gao (see [4]) established an improvement of the inequality (1.1)

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} \le \{\sum_{n=0}^{\infty}\omega(n)a_n^2\}^{1/2}\{\sum_{n=0}^{\infty}\omega(n)b_n^2\}^{1/2}, \tag{1.3}$$

where $\omega(n) = \pi - \theta(n)/\sqrt{2n+1}$ with $\theta(n) > 0$ $(n = 0, 1, 2\cdots)$. And then Gao(see [5]) applied Euler-Maclaurin summation formula to estimate the weight function $\omega(n) \le \pi - \theta/\sqrt{2n+1}$ for $\theta = 17/20$.

At the same time, the integral inequality was given

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y+1}dxdy \le \{\int_0^{\infty}\omega(x)f^2(x)dx\int_0^{\infty}\omega(x)g^2(x)dx\}^{\frac{1}{2}}, \tag{1.4}$$

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where $\omega(x) = \pi - 2\arctan(1/\sqrt{2x+1})$. Two years later, Yang(see [6]) built the following inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+1}dxdy < \pi\{\int_0^\infty \omega(x)f^2(x)dx \int_0^\infty \omega(x)g^2(x)dx\}^{\frac{1}{2}}, \quad (1.5)$$

where $\omega(x) = 1 - \frac{1-2/\pi}{(x+1)^{1/2}}$ and the constant factor π is the best possible.

In 2003, Yang Xiaojing (see [9]) got the following result

$$\int_0^\infty \int_0^\infty \frac{G(x)G(y)}{x+y+1}dxdy \leq \{\int_0^\infty (\frac{\pi}{\sin(\pi/p)} - r(x))F^p(x)dx\}^{\frac{1}{p}}$$

$$\times\{\int_0^\infty (\frac{\pi}{\sin(\pi/p)} - s(x))G^q(x)dx\}^{\frac{1}{q}} \quad (1.6)$$

where

$$r(x) = \int_0^{1/(px+p-1)} \frac{dt}{t^{1/p}(1+t)}, x \geq 0;\ s(x) = \int_0^{(p-1)/(px+1)} \frac{dt}{t^{1/p}(1+t)}, x \geq 0.$$

The equality holds if and only if $F(x) \equiv 0$ or $G(x) \equiv 0$.

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{m+n+1} \leq \pi\{\sum_{n=0}^\infty \alpha(n)a_n^p\}^{1/p}\{\sum_{n=0}^\infty \beta(n)b_n^q\}^{1/q} \quad (1.7)$$

where $\alpha(n) = \frac{\pi}{\sin(\pi/p)} - r(n), \beta(n) = \frac{\pi}{\sin(\pi/p)} - s(n)$ and $r(n) > 0, s(n) > 0$. The equality holds if and only if (a_n) or (b_n) is null.

2. Main results and applications

Theorem 2.1. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f, g > 0$ such that $(x+1)^{p-1}f^p(x) \in L^p(0,\infty)$, $(x+1)^{q-1}g^q(x) \in L^q(0,\infty)$, then we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1}dxdy$$

$$< \frac{\pi}{\sin(\pi/p)}\{\int_0^\infty \omega(x,p)f^p(x)dx\}^{1/p}\{\int_0^\infty \omega(x,q)g^q(x)dx\}^{1/q} \quad (2.1)$$

where $\omega(x,r) = [1 - \frac{1-\frac{s\sin(\pi/r)}{\pi}}{(\ln(x+1)+1)^{1/r}}](x+1)^{r-1}$ for $r = p, q, s = \frac{r}{r-1}$ and the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

In particular, for $p = q = 2$, we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1}dxdy$$

$$< \pi\{\int_0^\infty [1-\frac{1-2/\pi}{(\ln(x+1)+1)^{1/2}}](x+1)f^2(x)dx \int_0^\infty [1-\frac{1-2/\pi}{(\ln(x+1)+1)^{1/2}}](x+1)g^2(x)dx\}^{\frac{1}{2}} \quad (2.1a)$$

Proof. Setting $f(x) = \frac{1}{\sqrt[q]{x+1}}F(x), g(y) = \frac{1}{\sqrt[p]{y+1}}G(y)$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1}dxdy \\
= &\int_0^\infty \int_0^\infty \frac{1}{\ln(x+1)+\ln(y+1)+1} \\
&\{f(x)[\frac{\ln(x+1)+1}{\ln(y+1)+1}]^{1/pq}\}\{g(y)[\frac{\ln(y+1)+1}{\ln(x+1)+1}]^{1/pq}\}dxdy \\
= &\int_0^\infty \int_0^\infty \frac{1}{\ln(x+1)+\ln(y+1)+1}\{\frac{1}{\sqrt[p]{y+1}}F(x)[\frac{\ln(x+1)+1}{\ln(y+1)+1}]^{1/pq}\} \\
&\{\frac{1}{\sqrt[q]{x+1}}G(y)[\frac{\ln(y+1)+1}{\ln(x+1)+1}]^{1/pq}\}dxdy \\
\leq &\{\int_0^\infty \int_0^\infty \frac{\frac{1}{y+1}F^p(x)}{\ln(x+1)+\ln(y+1)+1}[\frac{\ln(x+1)+1}{\ln(y+1)+1}]^{1/q}dxdy\}^{1/p} \\
&\{\int_0^\infty \int_0^\infty \frac{\frac{1}{x+1}G^q(y)}{\ln(x+1)+\ln(y+1)+1}[\frac{\ln(y+1)+1}{\ln(x+1)+1}]^{1/p}dxdy\}^{1/q} \\
= &\{\int_0^\infty [\int_0^\infty \frac{\frac{1}{y+1}}{\ln(x+1)+\ln(y+1)+1}(\frac{\ln(x+1)+1}{\ln(y+1)+1})^{1/q}dy]F^p(x)dx\}^{1/p} \\
&\{\int_0^\infty [\int_0^\infty \frac{\frac{1}{x+1}}{\ln(x+1)+\ln(y+1)+1}(\frac{\ln(y+1)+1}{\ln(x+1)+1})^{1/p}dx]G^q(y)dy\}^{1/q}
\end{aligned}$$

Setting

$$\int_0^\infty \frac{\frac{1}{y+1}}{\ln(x+1)+\ln(y+1)+1}(\frac{\ln(x+1)+1}{\ln(y+1)+1})^{1/q}dy = \frac{\pi}{\sin(\pi/p)} - \frac{\theta(x,q)}{[\ln(x+1)+1]^{1/p}}$$

i.e.

$$\theta(x,q) = \frac{\pi}{\sin(\pi/p)}[\ln(x+1)+1]^{1/p} - \int_0^\infty \frac{\frac{1}{y+1}[\ln(x+1)+1]}{\ln(x+1)+\ln(y+1)+1}(\frac{1}{\ln(y+1)+1})^{1/q}dy$$

Since

$$\frac{\pi}{\sin(\pi/p)} = \int_0^\infty \frac{1}{(1+t)t^{1/q}}dt = \int_0^\infty \frac{\frac{1}{y+1}[\ln(x+1)+1]^{1/q}}{[\ln(x+1)+\ln(y+1)+1][\ln(y+1)]^{1/q}}dy$$

We have

$$\theta(x,q) = \int_0^\infty \frac{\frac{1}{y+1}[\ln(x+1)+1]}{\ln(x+1)+\ln(y+1)+1}[(\frac{1}{\ln(y+1)})^{1/q} - (\frac{1}{\ln(y+1)+1})^{1/q}]dy$$

$$\triangleq \int_0^\infty F(x,y)dy$$

$$\theta'(x,q) = \int_0^\infty \frac{\partial}{\partial x} F(x,y)dy$$
$$= \int_0^\infty \frac{\frac{1}{(y+1)(x+1)}\ln(y+1)}{[\ln(x+1)+\ln(y+1)+1]^2}[(\frac{1}{\ln(y+1)})^{1/q} - (\frac{1}{\ln(y+1)+1})^{1/q}]dy > 0$$

It means that $\theta(x,q)$ is a strictly increasing function on $[0,\infty)$, so

$$\theta(x,q) \geq \theta(0) = \int_0^\infty \frac{\frac{1}{y+1}}{\ln(y+1)+1}[(\frac{1}{\ln(y+1)})^{1/q} - (\frac{1}{\ln(y+1)+1})^{1/q}]dy = \frac{\pi}{\sin(\pi/p)} - q$$

The equality holds if and only if $x = 0$.

In the same way,

$$\theta(y,p) = \frac{\pi}{\sin(\pi/p)}[\ln(y+1)+1]^{1/q} - \int_0^\infty \frac{\frac{1}{x+1}[\ln(y+1)+1]}{\ln(y+1)+\ln(x+1)+1}(\frac{1}{\ln(x+1)+1})^{1/p}dx$$
$$\geq \frac{\pi}{\sin(\pi/p)} - p$$

Hence for $x \in (0,\infty)$, we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1}dxdy$$
$$< \frac{\pi}{\sin(\pi/p)}\{\int_0^\infty [1 - \frac{1 - \frac{q\sin(\pi/p)}{\pi}}{(\ln(x+1)+1)^{1/p}}]F^p(x)dx\}^{1/p}$$
$$\times\{\int_0^\infty [1 - \frac{1 - \frac{p\sin(\pi/p)}{\pi}}{(\ln(x+1)+1)^{1/q}}]G^q(x)dx\}^{\frac{1}{q}}$$
$$= \frac{\pi}{\sin(\pi/p)}\{\int_0^\infty [1 - \frac{1 - \frac{q\sin(\pi/p)}{\pi}}{(\ln(x+1)+1)^{1/p}}](x+1)^{p-1}f^p(x)dx\}^{1/p}$$
$$\times\{\int_0^\infty [1 - \frac{1 - \frac{p\sin(\pi/p)}{\pi}}{(\ln(x+1)+1)^{1/q}}](x+1)^{q-1}g^q(x)dx\}^{\frac{1}{q}}$$

This proves inequality (2.1). It remains to show that the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. In fact, define the functions by

$$\tilde{f}(x) = 0, \quad x \in (0, \sqrt{10}-1); \quad \frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)/p}}{x+1}, \quad x \in [\sqrt{10}-1, \infty).$$

$$\tilde{g}(y) = 0, \quad y \in (0, \sqrt{10}-1); \quad \frac{[\ln(y+1)+\frac{1}{2}]^{-(1+\varepsilon)/q}}{y+1}, \quad y \in [\sqrt{10}-1, \infty).$$

Where ε is small and positive. Assume that $\frac{\pi}{\sin(\pi/p)}$ is not the best possible, then there exist $K>0, K<\frac{\pi}{\sin(\pi/p)}$, such that

$$\int_0^\infty\int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{\ln(x+1)+\ln(y+1)+1}dxdy$$

$$< K\{\int_0^\infty[1-\frac{1-\frac{q\sin(\pi/p)}{\pi}}{(\ln(x+1)+1)^{1/p}}](x+1)^{p-1}\tilde{f}^p(x)dx\}^{1/p}$$

$$\times\{\int_0^\infty[1-\frac{1-\frac{p\sin(\pi/p)}{\pi}}{(\ln(x+1)+1)^{1/q}}](x+1)^{q-1}\tilde{g}^q(x)dx\}^{\frac{1}{q}}\le K/\varepsilon$$

On the other hand, for $x\ge\sqrt{10}-1$

$$\begin{aligned}
&\int_0^{\frac{1}{\ln(x+1)+\frac{1}{2}}}\frac{1}{1+t}t^{-(1+\varepsilon)/q}dt\\
=\ & \frac{1}{1-\frac{1+\varepsilon}{q}}[\frac{1}{\ln(x+1)+\frac{1}{2}}]^{1-\frac{1+\varepsilon}{q}}\\
\le\ & 2p\ [\frac{1}{\ln(x+1)+\frac{1}{2}}]^{1/2p} \qquad (assume\ 0<\varepsilon<\frac{q}{2p})
\end{aligned}$$

Hence

$$\begin{aligned}
& 0<\int_{\sqrt{10}-1}^\infty\frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)}}{x+1}\int_0^{\frac{1}{\ln(x+1)+\frac{1}{2}}}\frac{1}{1+t}t^{-(1+\varepsilon)/q}dtdx\\
\le\ & 2p\int_{\sqrt{10}-1}^\infty\frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)}}{x+1}[\frac{1}{\ln(x+1)+\frac{1}{2}}]^{1/2p}dx\\
=\ & 2p\int_{\sqrt{10}-1}^\infty\frac{[\ln(x+1)+\frac{1}{2}]^{-(\frac{1}{2p}+1+\varepsilon)}}{x+1}dx\\
\le\ & 2p\int_{\sqrt{10}-1}^\infty\frac{[\ln(x+1)+\frac{1}{2}]^{-\frac{1}{2p}-1}}{x+1}dx\\
=\ & 4p^2
\end{aligned}$$

This shows that

$$\int_{\sqrt{10}-1}^\infty\frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)}}{x+1}\int_0^{\frac{1}{\ln(x+1)+\frac{1}{2}}}\frac{1}{1+t}t^{-(1+\varepsilon)/q}dtdx=O(1)$$

Then we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{\ln(x+1)+\ln(y+1)+1}dxdy\\
= &\int_{\sqrt{10}-1}^\infty \frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)/p}}{x+1}\{\int_{\sqrt{10}-1}^\infty \frac{[\ln(y+1)+\frac{1}{2}]^{-(1+\varepsilon)/q}}{\ln(x+1)+\ln(y+1)+1}\frac{1}{y+1}dy\}dx\\
= &\int_{\sqrt{10}-1}^\infty \frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)}}{x+1}\{\int_{\frac{1}{\ln(x+1)+\frac{1}{2}}}^\infty \frac{1}{1+t}t^{-(1+\varepsilon)/q}dt\}dx\\
= &\int_{\sqrt{10}-1}^\infty \frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)}}{x+1}\{\int_0^\infty \frac{1}{1+t}t^{-(1+\varepsilon)/q}dt\}dx\\
&-\int_{\sqrt{10}-1}^\infty \frac{[\ln(x+1)+\frac{1}{2}]^{-(1+\varepsilon)}}{x+1}\{\int_0^{\frac{1}{\ln(x+1)+\frac{1}{2}}} \frac{1}{1+t}t^{-(1+\varepsilon)/q}dt\}dx\\
= &\frac{1}{\varepsilon}[\frac{\pi}{\sin(\pi/p)}+\circ(1)]-O(1)\\
= &\frac{1}{\varepsilon}[\frac{\pi}{\sin(\pi/p)}+\circ(1)]
\end{aligned}$$

Thus $\frac{1}{\varepsilon}[\frac{\pi}{\sin(\pi/p)}+\circ(1)]\le K/\varepsilon$. i.e. $\frac{\pi}{\sin(\pi/p)}+\circ(1)\le K$, or $\frac{\pi}{\sin(\pi/p)}\le K$, when ε is sufficiently small. Which is in contraction with the assumption. This proves that the constant factor in (2.1) is the best possible. □

Corollary 1 If $f,g(f,g\neq 0)$ are real functions such that $(x+1)f^2(x)\in L^2(0,\infty)$, $(x+1)g^2(x)\in L^2(0,\infty)$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+3}dxdy$$

$$<\pi\{\int_0^\infty[1-\frac{1-2/\pi}{(\ln(x+1)+1)^{1/2}}](x+1)f^2(x)dx\int_0^\infty[1-\frac{1-2/\pi}{(\ln(x+1)+1)^{1/2}}](x+1)g^2(x)dx\}^{\frac{1}{2}}$$

Proof. Notice that $\ln(x+1)\le x+1, \ln(y+1)\le y+1$, we get the result consequently. □

Theorem 2.2. *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f>0$ such that $(x+1)^{p-1}f^p(x)\in L^p(0,\infty)$, then we have*

$$\int_0^\infty(\omega(y,q))^{1-p}[\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx]^p dy<[\frac{\pi}{\sin(\pi/p)}]^p\int_0^\infty \omega(x,p)f^p(x)dx \tag{2.2}$$

$$\int_0^\infty \frac{1}{y+1}[\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx]^p<[\frac{\pi}{\sin(\pi/p)}]^p\int_0^\infty \omega(x,p)f^p(x)dx \tag{2.3}$$

where the constant factor $[\frac{\pi}{\sin(\pi/p)}]^p$ is the best possible for the two inequalities and inequality (2.2) is equivalent to (2.1).

Proof. We set

$$g(y) := (\omega(y,q))^{1-p}[\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx]^{p-1},$$

then by (2.1), we have

$$\begin{aligned} 0 &< \int_0^\infty \omega(y,q)g^q(y)dy \\ &= \int_0^\infty (\omega(y,q))^{1-p}[\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx]^p dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1}dxdy \\ &\leq \frac{\pi}{\sin(\pi/p)}\{\int_0^\infty \omega(x,p)f^p(x)dx\}^{1/p}\{\int_0^\infty \omega(y,q)g^q(y)dy\}^{\frac{1}{q}} \end{aligned} \tag{2.4}$$

Hence we obtain

$$\begin{aligned} 0 &< \{\int_0^\infty \omega(y,q)g^q(y)dy\}^{1/q} \\ &= \{\int_0^\infty (\omega(y,q))^{1-p}[\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx]^p dy\}^{1/p} \\ &\leq \frac{\pi}{\sin(\pi/p)}\{\int_0^\infty \omega(x,p)f^p(x)dx\}^{1/p} < \infty \end{aligned} \tag{2.5}$$

By (2.1), both (2.4) and (2.5) take the form of strict inequality, and then we get (2.2).

On the other hand, assume that (2.2) is valid. By Hölder's inequality, we obtain

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1}dxdy \\ = &\int_0^\infty [(\omega(y,q))^{-1/q}\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx][(\omega(y,q))^{1/q}g(y)dy] \\ \leq &\{\int_0^\infty [(\omega(y,q))^{1-p}\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1}dx]^p dy\}^{\frac{1}{p}}\{\int_0^\infty \omega(x,q)g^q(x)dx\}^{\frac{1}{q}} \end{aligned} \tag{2.6}$$

Then by (2.2), we have (2.1). Thus (2.1) and (2.2) are equivalent.

If the constant factor $[\frac{\pi}{\sin(\pi/p)}]^p$ in (2.2) is not the best possible, using (2.6), we may get a contradiction that the constant factor in (2.1) is not the best possible.

As for (2.3), we can get it from (2.2) right away. Similarly, the constant factor $[\frac{\pi}{\sin(\pi/p)}]^p$ in (2.3) is also the best possible. This completes the proof. □

Corollary 2 Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f, g > 0$ such that $(x+1)^{p-1} f^p(x) \in L^p(0, \infty)$, $(x+1)^{q-1} g^q(x) \in L^q(0, \infty)$, then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\ln(x+1)+\ln(y+1)+1} dxdy$$
$$< \frac{\pi}{\sin(\pi/p)} \{\int_0^\infty (x+1)^{p-1} f^p(x)dx\}^{1/p} \{\int_0^\infty (y+1)^{q-1} g^q(y)dy\}^{\frac{1}{q}} \tag{2.7}$$

$$\int_0^\infty \frac{1}{y+1} [\int_0^\infty \frac{f(x)}{\ln(x+1)+\ln(y+1)+1} dx]^p dy < [\frac{\pi}{\sin(\pi/p)}]^p \int_0^\infty (x+1)^{p-1} f^p(x)dx \tag{2.8}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ and $[\frac{\pi}{\sin(\pi/p)}]^p$ are both the best possible and the two inequalities are equivalent.

Proof. The proof can be completed by following the similar steps as in the proof of Theorem 2.2. Thus we omit the details. □

Remark 2.1 (i) (2.1) is a new Hardy-Hilbert's type inequality. Obviously, it is a refinement of (2.7).
(ii) (2.8) is also a new Hilbert's type inequality which comes from (2.3), in other words , (2.3) is an improvement of (2.8).

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