SOME IDENTITIES INVOLVING COMPLEX POLYNOMIALS

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Abstract. In this paper a method to obtain identities in the complex plane is presented. Using analytic techniques from complex analysis several identities for polynomials are obtained. The results given here are rational identities involving finite sums of zeros and coefficients of complex polynomials.

1. Introduction

The problem of finding relations between the zeroes and coefficients of a polynomial occupies a central role in the theory of equations. The most well known of such relations are Cardan-Viète's formulae. Namely, if the complex polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, has zeros $z_1, z_2, \cdots, z_n$, then

$$\sum_{1\le i_1<i_2<\cdots<i_k\le n} z_{i_1} z_{i_2} \cdots z_{i_k} = (-1)^k \, \frac{a_{n-k}}{a_n}, \quad 1 \le k \le n \tag{1.1}$$

In this paper, taking into account the preceding relations and using complex variable techniques like the ones employed by H. W. Gould in 1956 to evaluate Stirling numbers ([1], [2]), later by G. Egorychev in 1977 [3] to obtain combinatorial identites and recently by others to get rational sums [4], some rational zero and coefficient identities involving complex polynomials similar to the ones given in ([5], [6]) are obtained.

2. Main results

In the sequel some rational identities are given. The first one, that it is an immediate consequence of the Fundamental Theorem of Algebra, it is stated and proved in the following

Theorem 1. Suppose that the distinct nonzero complex numbers $z_1, z_2, \cdots, z_n$ are the zeros of the complex polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$, ($a_k \neq 0$ for $0 \le k \le n$), then holds

$$\frac{1}{a_0} + \sum_{k=1}^{n} \frac{1}{z_k A'(z_k)} = 0.$$

Proof. Let $B(z)$ be the complex polynomial defined by

$$B(z) = -1 + \sum_{k=1}^{n} \prod_{j\neq k} \frac{z - z_j}{z_k - z_j}.$$

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Since $B(z)$ has degree $n-1$ and $B(z_k)=0$ for $1 \le k \le n$, then, by the Fundamental Theorem of Algebra, $B(z)$ is identically zero. Therefore,

$$1 = 1 + B(0) = \sum_{k=1}^{n} \prod_{j \neq k} \frac{-z_j}{z_k - z_j} = (-1)^{n-1} z_1 z_2 \cdots z_n \sum_{k=1}^{n} \frac{1}{z_k} \prod_{j \neq k} \frac{1}{z_k - z_j}$$

or

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{j \neq k} \frac{1}{z_k - z_j} = \frac{(-1)^{n-1}}{z_1 z_2 \cdots z_n}$$

From the preceding and taking into account Cardan-Viète formulae, we get

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{j \neq k} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{a_n}{z_k A'(z_k)} = \frac{(-1)^{n-1} a_n}{(-1)^n a_0}$$

from which the statement immediately follows and the result is proven.

□

In what follows the results presented show how to apply complex variable techniques to generate new identities.

Theorem 2. Let $A(z) = \sum_{k=0}^{n} a_k z^k$, ($a_k \neq 0$ for $0 \le k \le n$), be a polynomial with complex coefficients having simple zeroes $z_1, z_2, \cdots, z_n$. Then holds

$$\sum_{k=0}^{n} \frac{z_k^{n-1}}{A'(z_k)} = \frac{1}{a_n}.$$

Proof. Consider the integral $I = \dfrac{1}{2\pi i} \oint_\gamma \dfrac{z_k^{n-1}}{A(z)}\, dz$ where γ is a circle centered at the origin and radius $r > \max\limits_{1 \le k \le n} \{|z_k|\}$. Integrating inside γ contour, we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi i} \oint_\gamma \frac{z^{n-1}}{A(z)}\, dz = \sum_{k=1}^{n} \operatorname{Res}\left(\frac{z^{n-1}}{A(z)}, z = z_k \right) \\ &= \frac{1}{a_n} \sum_{k=1}^{n} z_k^{n-1} \prod_{j \neq k} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{z_k^{n-1}}{A'(z_k)}. \end{aligned}$$

On the other hand, evaluating the integral outside γ contour, we have

$$I_2 = \frac{1}{2\pi i} \oint_\gamma \frac{z^{n-1}}{A(z)}\, dz = \operatorname{Res}\left(\frac{z^{n-1}}{A(z)}, z = \infty \right) = \operatorname{Res}\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right), z = 0 \right)$$

where $f(z) = \dfrac{z^{n-1}}{A(z)}$. Therefore,

$$\begin{aligned} I_2 &= \operatorname{Res}\left(-\frac{1}{z^2} \frac{1}{z^{n-1} A(1/z)}, z = 0 \right) = -\operatorname{Res}\left(\frac{1}{z^{n+1} A(1/z)}, z = 0 \right) \\ &= -\operatorname{Res}\left(\frac{1}{z(a_n + a_{n-1} z + \cdots + a_0 z^n)}, z = 0 \right) = -\frac{1}{a_n}. \end{aligned}$$

Applying Cauchy's theorem on contour integrals, we have $I_1 + I_2 = 0$ and we are done.

□

Theorem 3. Suppose that the distinct nonzero complex numbers $z_1, z_2, \cdots, z_n$ are the zeros of the polynomial with complex coefficients $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$. Then, for all $\ell \geq 3$, holds

$$\sum_{k=1}^{n} \frac{1+z_k^{\ell-1}}{z_k^2 A'(z_k)} = \sum_{k=1}^{n} \frac{1}{z_k} \prod_{j=1}^{n} \frac{1}{z_j} \tag{2.1}$$

Proof. To prove (2.1) we will evaluate the integral $I = \dfrac{1}{2\pi i} \oint_\gamma \dfrac{1+z^{\ell-1}}{z^2 A_n(z)}\, dz$ over the interior and exterior domains limited by γ, a circle centered at the origin and radius $r < \min\limits_{1\leq k\leq n} \Big\{ |z_k| \Big\}$. Integrating in the region outside of γ contour we have

$$\begin{aligned} I_1 = \frac{1}{2\pi i} \oint_\gamma \frac{z^{\ell-1}+1}{z^2 A_n(z)}\, dz &= \sum_{k=1}^{n} \mathrm{Res}\left\{ \frac{z^{\ell-1}+1}{z^2 A_n(z)}\ ,\ z = z_k \right\} \\ &= \sum_{k=1}^{n} \frac{1+z_k^{\ell-1}}{z_k^2} \prod_{j\neq k} \frac{1}{z_k - z_j} \\ &= (-1)^{n-1} \sum_{k=1}^{n} \frac{1+z_k^{\ell-1}}{z_k^2} \prod_{j\neq k} \frac{1}{z_j - z_k} \\ &= (-1)^{n-1} \sum_{k=1}^{n} \frac{1+z_k^{\ell-1}}{z_k^2 A'(z_k)}. \end{aligned}$$

Integrating in the region inside of γ contour and taking into account (1.1), we get

$$\begin{aligned} I_2 = \frac{1}{2\pi i} \oint_\gamma \frac{z^{\ell-1}+1}{z^2 A_n(z)}\, dz &= \mathrm{Res}\left\{ \frac{z^{\ell-1}+1}{z^2 A_n(z)}\ ,\ z = 0 \right\} \\ &= \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz} \left\{ \frac{z^{\ell-1}+1}{A_n(z)} \right\} \\ &= -\frac{A_n'(0)}{A_n^2(0)} = -\frac{a_1}{a_0^2} \\ &= -(-1)^{n-1} \sum_{k=1}^{n} \frac{1}{z_k} \prod_{j=1}^{n} \frac{1}{z_j}. \end{aligned}$$

Applying again Cauchy's result on contour integrals, we have $I_1 + I_2 = 0$ and the proof is complete.

□

The same procedure can be applied to obtain many nice identities. In the sequel, we give several zero identities that have been obtained applying this technique. We begin with

Theorem 4. Suppose that $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ is a polynomial of degree n with $n \geq 2$, $A(z)$ has n distinct zeros $z_1, \ldots, z_n$, and ζ is a complex number such that no ratio of two zeros of $A(z)$ is equal to ζ. Then,

$$\sum_{k=1}^{n} \frac{1}{A'(z_k)} \left(\frac{1}{A(\zeta z_k)} + \frac{1}{\zeta A(z_k/\zeta)} \right) = 0.$$

Proof. We will evaluate the integral $I = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{1}{A(z)A(\zeta z)}\,dz$ over the interior and exterior domains limited by γ, where $\gamma = \{z \in \mathbb{C} \mid |z| < r\}$ with $\displaystyle\min_{1\le j\le n}\Big\{|z_k|\Big\} > r > \max_{1\le j\le n}\Big\{|z_k/\zeta|\Big\}$. Integrating in the region outside of γ contour we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta z)}\,dz = \sum_{k=1}^{n} \operatorname{Res}\left\{\frac{1}{A(z)A(\zeta z)}\ ,\ z = z_k\right\} \\ &= \sum_{k=1}^{n} \frac{1}{A(\zeta z_k)} \prod_{j\neq k}^{n} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{1}{A(\zeta z_k)A'(z_k)}. \end{aligned}$$

Integrating in the region inside of γ contour we get

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\oint_\gamma \frac{1}{A(z)A(\zeta z)}\,dz = \sum_{k=1}^{n} \operatorname{Res}\left\{\frac{1}{A(z)A(\zeta z)}\ ,\ z = z_k/\zeta\right\} \\ &= \sum_{k=1}^{n} \frac{1}{\zeta A(z_k/\zeta)} \prod_{j\neq k}^{n} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{1}{\zeta A(z_k/\zeta)A'(z_k)}. \end{aligned}$$

Taking into account Cauchy's result on contour integrals, we have $I_1 + I_2 = 0$. This completes the proof.

□

Now we will give a generalization of the preceding result from one complex number to two complex numbers.

Theorem 5. Assume that the zeroes $z_1, z_2, \ldots, z_n$ of polynomial $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ are distinct nonzero complex numbers. Let ζ_1 and ζ_2 be complex numbers such that $\displaystyle\max_{1\le k\le n}\Big\{|z_k/\zeta_2|\Big\} < \min_{1\le k\le n}\Big\{|z_k/\zeta_1|\Big\}$. Then, holds

$$\sum_{k=1}^{n} \frac{1}{A'(z_k)} \left(\frac{A(\zeta_1 z_k)}{\zeta_2 A(z_k\zeta_1/\zeta_2)} + \frac{A(\zeta_2 z_k)}{\zeta_1 A(z_k\zeta_2/\zeta_1)} \right) = 0.$$

Proof. We will evaluate the integral $I = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}\,dz$ over the interior and exterior domains limited by γ, a circle centered at the origin and radius r such that $\displaystyle\max_{1\le k\le n}\Big\{|z_k/\zeta_2|\Big\} < r < \min_{1\le k\le n}\Big\{|z_k/\zeta_1|\Big\}$. Integrating in the region outside of γ contour we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}\,dz = \sum_{k=1}^{n} \operatorname{Res}\left\{\frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}\ ,\ z = z_k/\zeta_2\right\} \\ &= \sum_{k=1}^{n} \frac{A(\zeta_1 z_k)}{\zeta_2 A(z_k\zeta_1/\zeta_2)} \prod_{j\neq k} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{1}{A'(z)} \left(\frac{A(\zeta_1 z_k)}{\zeta_2 A(z_k\zeta_1/\zeta_2)} \right). \end{aligned}$$

Integrating in the region inside of γ contour we get

$$
\begin{aligned}
I_2 &= \frac{1}{2\pi i}\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}\,dz = \sum_{k=1}^{n} \operatorname{Res}\left\{\frac{A(\zeta_1\zeta_2 z)}{A(\zeta_1 z)A(\zeta_2 z)}\ ,\ z = z_k/\zeta_1\right\} \\
&= \sum_{k=1}^{n} \frac{A(\zeta_2 z_k)}{\zeta_1 A(z_k\zeta_2/\zeta_1)} \prod_{j\neq k} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{1}{A'(z)}\left(\frac{A(\zeta_2 z_k)}{\zeta_1 A(z_k\zeta_2/\zeta_1)}\right).
\end{aligned}
$$

According to Cauchy's result on contour integrals, we have $I_1 + I_2 = 0$ and the identity is proven.

□

Finally, evaluating the rational integrals $I = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{A(z)}{A(\zeta_1 z)A(\zeta_2 z)}\,dz$, $J = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(z)A(\zeta_1 z)A(\zeta_2 z)}\,dz$ and $K = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{A(\zeta_1\zeta_2 z)}{A(z)A(\zeta_1 z)A(\zeta_2 z)}\,dz$ over the interior and exterior domains limited by γ, a circle centered at the origin and radius r such that $\max\limits_{1\le k\le n}\left\{|z_k/\zeta_2|\right\} < r < \min\limits_{1\le k\le n}\left\{|z_k/\zeta_1|\right\}$, we have the following corollaries.

Corollary 6. Assume that the zeroes $z_1, z_2, \ldots, z_n$ of polynomial $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ are distinct nonzero complex numbers. Let ζ_1 and ζ_2 be complex numbers such that $\max\limits_{1\le k\le n}\left\{|z_k/\zeta_2|\right\} < \min\limits_{1\le k\le n}\left\{|z_k/\zeta_1|\right\}$. Then, holds

$$
\sum_{k=1}^{n} \frac{1}{A'(z)}\left(\frac{A(z_k/\zeta_1)}{\zeta_1 A(z_k\zeta_2/\zeta_1)} + \frac{A(z_k/\zeta_2)}{\zeta_2 A(z_k\zeta_1/\zeta_2)}\right) = 0.
$$

Corollary 7. Assume that the zeroes $z_1, z_2, \ldots, z_n$ of polynomial $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ are distinct nonzero complex numbers. Let ζ_1 and ζ_2 be complex numbers such that $\max\limits_{1\le k\le n}\left\{|z_k/\zeta_1|\right\} < \min\limits_{1\le k\le n}\left\{|z_k|\right\}$ and $\max\limits_{1\le k\le n}\left\{|z_k/\zeta_2|\right\} < \min\limits_{1\le k\le n}\left\{|z_k/\zeta_1|\right\}$. Then, holds

$$
\sum_{k=1}^{n} \frac{1}{A'(z)}\left(\frac{A(\zeta_1 z_k)}{\zeta_2 A(z_k/\zeta_2)A(z_k\zeta_1/\zeta_2)} + \frac{A(\zeta_2 z_k)}{\zeta_1 A(z_k/\zeta_1)A(z_k\zeta_2/\zeta_1)}\right) = 0
$$

and

$$
\sum_{k=1}^{n} \frac{1}{A'(z)}\left(\frac{A(\zeta_1\zeta_2 z_k)}{A(z_k\zeta_2)A(z_k\zeta_1)} + \frac{A(\zeta_1 z_k)}{\zeta_2 A(z_k/\zeta_2)A(z_k\zeta_1/\zeta_2)} + \frac{A(\zeta_2 z_k)}{\zeta_1 A(z_k/\zeta_1)A(z_k\zeta_2/\zeta_1)}\right) = 0.
$$

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