SOME APPLICATIONS OF AN INEQUALITY OF NORMAN LEVINSON

J. GIBERGANS-BÁGUENA, J. J. EGOZCUE AND J. L. DÍAZ-BARRERO

Abstract. Applying the inequality of N. Levinson several multivariate inequalities involving arithmetics progressions, random variables and binomial coefficients are obtained.

1. Introduction

In [1] Norman Levinson presented the following generalization of an inequality of Ky Fan [2]

Theorem 1.1. *Let* $f : [0, 2s] \to \mathbb{R}$ *be a function that with a nonnegative third derivative in* $(0, 2s)$. *If* $a_k \in (0, s], (1 \le k \le n)$, *and* $p_k > 0, (1 \le k \le n)$, *then*

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k f(x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right)$$

$$\le \frac{1}{P_n}\sum_{k=1}^{n} p_k f(2s - x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k (2s - x_k)\right) \tag{1.1}$$

where $P_j = \sum_{k=1}^{j} p_k$ *for* $1 \le j \le n$. *If* $f'''(t) > 0$ *in* $(0, 2s)$ *then equality holds if and only if* $x_1 = x_2 = \ldots = x_n$.

An immediate consequence of the preceding result is the above mentioned inequality of Ky Fan. Namely,

Corollary 1.2. *Let* $0 < x_k \le 1/2$ *for* $k = 1, 2, \cdots, n$. *Then*

$$\left(\prod_{k=1}^{n} x_k\right) \Bigg/ \left(\sum_{k=1}^{n} x_k\right)^n \le \left(\prod_{k=1}^{n} (1 - x_k)\right) \Bigg/ \left(\sum_{k=1}^{n} (1 - x_k)\right)^n \tag{1.2}$$

Proof. Indeed inequality (1.2) immediately follows setting $s = 1/2$ and $f(t) = \log(t)$ into (1.1). □

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2. The Inequalities

In the sequel some applications of the preceding inequality are given and several inequalities similar to the ones presented in ([3], [4]) are obtained. We begin with

Theorem 2.1. *Let $x_1, x_2, \dots, x_n$ be positive numbers such that $x_1 + x_2 + \cdots + x_n = s$ and let $p_k > 0, (1 \le k \le n)$ and $q \in \mathbb{N}^*$. Then holds*

$$\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k(2s - x_k)\right)^q + \frac{1}{P_n}\sum_{k=1}^{n} p_k x_k^q \le \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right)^q + \frac{1}{P_n}\sum_{k=1}^{n} p_k(2s - x_k)^q$$

Proof. Since the function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(t) = t^q, q \in \mathbb{N}^*$, has third derivative $f'''(t) \ge 0$, then applying (1.1) the statement immediately follows after rearranging terms and this completes the proof. □

Applying the preceding result the following corollaries are obtained.

Corollary 2.2. *Let $x_1, x_2, \dots, x_n$ be positive numbers in arithmetic progression and let $p_k > 0, (1 \le k \le n)$ such that $p_1 + p_2 + \cdots + p_n = 1$. Then, for all $q \in \mathbb{N}^*$, holds*

$$\left(\sum_{k=1}^{n} p_k\left(x_1 + x_n - \frac{x_k}{n}\right)\right)^q - \sum_{k=1}^{n} p_k\left(x_1 + x_n - \frac{x_k}{n}\right)^q \le \frac{1}{n^q}\left[\left(\sum_{k=1}^{n} p_k x_k\right)^q - \sum_{k=1}^{n} p_k x_k^q\right]$$

Corollary 2.3. *Let X be a Binomial random variable. That is, $X \sim B(n, p)$. Then, for all $q \in \mathbb{N}^*$, holds*

$$\left(\sum_{k=1}^{n} \frac{p_k(n(n+1) - k)}{1 - (1-p)^n}\right)^q + \sum_{k=1}^{n} \frac{p_k k^q}{1 - (1-p)^n} \le \left(\frac{np}{1 - (1-p)^n}\right)^q + \sum_{k=1}^{n} \frac{p_k\big(n(n+1) - k\big)^q}{1 - (1-p)^n}$$

where $p_k = \binom{n}{k} p^k (1-p)^{n-k}, 0 \le k \le n$.

Proof. Setting $x_k = k, 1 \leq k \leq n$ and noting that $\sum_{k=1}^{n} p_k = 1-(1-p)^n$ and $\sum_{k=1}^{n} p_k x_k = E(X) = np$, then the statement immediately follows from Theorem 2.1.

□

Another application of Levinson's inequality is the following result.

Theorem 2.4. *Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1 + x_2 + \cdots + x_n = s$ and let $p_k > 0, (1 \leq k \leq n)$ such that $p_1+p_2+\cdots+p_n = 1$. Then holds*

$$\left(\sum_{k=1}^{n} p_k x_k\right) \prod_{k=1}^{n} \left(\frac{2s}{x_k} - 1\right)^{p_k} \geq \sum_{k=1}^{n} p_k(2s - x_k) \tag{2.1}$$

Proof. To proof the preceding statement we will consider the function $f(t) = \log(t)$ and applying (1.1), we have

$$\sum_{k=1}^{n} p_k \log(x_k) - \log\left(\sum_{k=1}^{n} p_k x_k\right)$$
$$\leq \sum_{k=1}^{n} p_k \log(2s - x_k) - \log\left(\frac{1}{P_n} \sum_{k=1}^{n} p_k(2s - x_k)\right)$$

Taking into account that $\sum_{k=1}^{n} \log(x_k)^{p_k} = \log\left(\prod_{k=1}^{n} {x_k}^{p_k}\right)$ we have

$$\log \prod_{k=1}^{n} {x_k}^{p_k} - \log\left(\sum_{k=1}^{n} p_k x_k\right) \leq \log \prod_{k=1}^{n} (2s - x_k)^{p_k} - \log\left(\sum_{k=1}^{n} p_k(2s - x_k)\right)$$

or equivalently,

$$\log\left(\frac{\prod_{k=1}^{n} {x_k}^{p_k}}{\sum_{k=1}^{n} p_k x_k}\right) \leq \log\left(\frac{\prod_{k=1}^{n} (2s - x_k)^{p_k}}{\sum_{k=1}^{n} p_k(2s - x_k)}\right)$$

Since $f(x) = \log(x)$ is injective, we obtain

$$\frac{\displaystyle\sum_{k=1}^{n} p_k(2s - x_k)}{\displaystyle\sum_{k=1}^{n} p_k x_k} \le \prod_{k=1}^{n} \left(\frac{2s - x_k}{x_k}\right)^{p_k}$$

and the proof is complete. □

Finally, we present two new multivariate inequalities involving binomial coefficients.

Corollary 2.5. *Let $x_0, x_1, \cdots, x_n$ be positive numbers such that $x_0 + x_1 + \cdots + x_n = s$. Then holds*

$$\left[\sum_{k=0}^{n} \binom{n}{k} (2s - x_k)\right]^{2n} \le \left[\sum_{k=0}^{n} \binom{n}{k} x_k\right]^{2^n} \prod_{k=0}^{n} \left(\frac{2s - x_k}{x_k}\right)^{\binom{n}{k}}$$

Proof. Setting $p_k = \dfrac{1}{2^n}\dbinom{n}{k}, 0 \le k \le n$ into the preceding result and taking into account the well known identity $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we have

$$\frac{\dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n} \binom{n}{k} (2s - x_k)}{\dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n} \binom{n}{k} x_k} \le \prod_{k=1}^{n} \left(\frac{2s - x_k}{x_k}\right)^{\frac{\binom{n}{k}}{2^n}}$$

or equivalently,

$$\left[\frac{\displaystyle\sum_{k=0}^{n} \binom{n}{k} (2s - x_k)}{\displaystyle\sum_{k=0}^{n} \binom{n}{k} x_k}\right]^{2^n} \le \prod_{k=0}^{n} \left(\frac{2s - x_k}{x_k}\right)^{\binom{n}{k}}$$

from which the statement immediately follows and this completes the proof. □

Corollary 2.6. *If $0 \le x_k \le \frac{1}{2}, 0 \le k \le n$ and $x_0 + x_1 + \cdots + x_n = s$. Then*

$$\left(\frac{1}{2^{n-1}} \sum_{k=0}^{n} \binom{n}{k} (2s - x_k)\right)^{2n} \le \prod_{k=0}^{n} \left(\frac{2s - x_k}{x_k}\right)^{\binom{n}{k}}$$

holds.

Proof. We have $\sum_{k=0}^{n}\binom{n}{k}x_k \le \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} = 2^{n-1}$ and applying (2.1) yields

$$\left[\frac{\sum_{k=0}^{n}\binom{n}{k}(2s-x_k)}{2^{n-1}}\right]^{2n} \le \prod_{k=0}^{n}\left(\frac{2s-x_k}{x_k}\right)^{\binom{n}{k}}$$

and the statement follows. □

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Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, 08034 Barcelona. Spain

E-mail address: jose.gibergans@upc.edu

E-mail address: juan.jose.egozcue@upc.edu

E-mail address: jose.luis.diaz@upc.edu