Minimum mass-radius ratio for charged general relativistic objects

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Abstract

We rigorously prove that for compact charged general relativistic objects there is a lower bound for the mass-radius ratio. This result follows from the same Buchdahl type inequality for charged objects, which has been extensively used for the proof of the existence of an upper bound for the mass-radius ratio. The effect of the vacuum energy (a cosmological constant) on the minimum mass is also taken into account. Several bounds on the total charge, mass and the vacuum energy for compact charged objects are obtained from the study of the Ricci scalar invariants. The total energy (including the gravitational one) and the stability of the objects with minimum mass-radius ratio is also considered, leading to a representation of the mass and radius of the charged objects with minimum mass-radius ratio in terms of the charge and vacuum energy only.

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I. INTRODUCTION

The bag models of hadrons [1], proposed in the 1970's, have had a remarkable phenomenological success (see [2] and [3] for reviews and recent developments). In these models, hadrons consist of free (or only weekly interacting) quarks, which are confined to a finite region of space, called the bag. The confinement is not a dynamical one, but it is put in by hand, imposing some appropriate boundary conditions. The bag is stabilized by a term of the form $g_{\mu\nu}B$, which is added to the energy momentum tensor $T_{\mu\nu}$ inside the bag, which thus takes the form $T_{\mu\nu} = T_{\mu\nu}^{(\text{fields})} + g_{\mu\nu}B$. By recalling the energy-momentum tensor of a perfect fluid in its rest-frame, $T_{\mu\nu} = \text{diag}\,(\epsilon, -p, -p, -p)$, where ϵ is the energy density and p is the thermodynamic pressure, it immediately follows that the bag constant B is immediately interpreted as positive contribution to the energy density ϵ and a negative contribution to the pressure p inside the bag. Equivalently, we may attribute a term $-g_{\mu\nu}B$ to the region outside the bag. This leads to a picture of a non-trivial vacuum with a negative energy density $\epsilon_{vac} = -B$ and a positive pressure $p_{vac} = +B$. The stability of the hadron then results from balancing this positive vacuum pressure with the pressure caused by the quarks inside the bag [3].

Therefore quark bag models in the theories of strong interactions assume that the breaking of physical vacuum takes place inside hadrons. As a result the vacuum energy densities inside and outside a hadron become essentially different and the vacuum pressure B on a bag wall equilibrates the pressure of quarks thus stabilizing the system. The MIT bag model says nothing about the origin of the non-trivial vacuum, but treats B as a free parameter. Assuming a static spherical bag of radius R, the mass of the hadron is given by the sum $E_{BM} = 4\pi BR^3/3 - z_0/R + \sum_q x_q/R + ...$, where the first term corresponds to the volume energy, required to replace the non-trivial vacuum by the trivial one inside the bag, the second term parameterize the finite part of the zero-point energy of the bag and the third term is the sum of the rest and kinetic energy of the quarks [3].

The finite electron self-energy is a puzzling problem in both quantum theory and classical theory. Quantum electrodynamics, with its remarkable predictive power, fails to explain the origin of the finite electron mass, and none of the proposed regularization schemes have succeeded in predicting the observed mass. On the other hand, a point charge is incompatible with classical electrodynamics, because it has the self-energy and stability problems. An electron of finite radius was proposed by Abraham and Lorentz, with the particle radius equal to $R = Q^2/M$, where Q and M are the charge and the mass of the particle, respectively. This relation has been obtained by assuming that the electromagnetic potential energy of the particle Q^2/R is equal to its mass M, according to the mass-energy equivalence law. However, an extended charge distribution interacting with itself cannot be stable and non-electromagnetic forces are needed to prevent the electron from exploding. Such cohesive non-electromagnetic forces were suggested by Poincaré, and are called Poncaré stresses [4].

On the other hand the Einstein-Maxwell field equations of general relativity can be used to construct a Lorentz model of an electron as an extended body consisting of pure charge and no matter and electromagnetic mass models for static spherically symmetric charged fluid distributions have been extensively studied [5]. The Poincaré stresses are explained as due to vacuum polarization, the vacuum energy density ρ_V and the vacuum pressure p_V satisfying an equation of state of the form $\rho_V + p_V = 0$, where in general the vacuum energy density $\rho_V > 0$ and the pressure $p_V < 0$. This type of equation of state implies that the matter distribution under consideration is in tension, in a state known as "false vacuum" or "degenerate vacuum". The gravitational blue-shift of light is explained as due to repulsive gravitation produced by the negative gravitational mass of the polarized vacuum. In the context of general relativity, the electron, modelled as a spherically symmetric charged distribution of matter, must contain some negative rest mass if its radius is not larger than 10^{-16} cm. In some extended electron models, the negative energy density distributions result from the requirement that the total mass of these models remains constant in the limit of a point particle.

The observations of high redshift supernovae [6] and the Boomerang/Maxima data [7], showing that the location of the first acoustic peak in the power spectrum of the microwave background radiation is consistent with the inflationary prediction $\Omega = 1$, have provided compelling evidence for a net equation of state of the cosmic fluid lying in the range $-1 \leq w = p/\rho < -1/3$. To explain these observations, two dark components are invoked: the pressure-less cold dark matter (CDM) and the dark energy (DE) with negative pressure. CDM contributes $\Omega_m \sim 0.25$, and is mainly motivated by the theoretical interpretation of the galactic rotation curves and large scale structure formation. DE provides $\Omega_{DE} \sim 0.7$ and is responsible for the acceleration of the distant type Ia supernovae. The best candidate for the dark energy is the cosmological constant Λ, which is usually interpreted physically as a vacuum energy. Its size is of the order $\Lambda \approx 3 \times 10^{-56}$ cm^{-2} [8].

By using the static spherically symmetric gravitational field equations Buchdahl [9] has obtained an absolute constraint of the maximally allowable mass M- radius R for isotropic fluid spheres of the form $2M/R \leq 8/9$ (where natural units $c = G = 1$ have been used).

The existence of the cosmological constant modifies the allowed ranges for various physical parameters, like, for example, the maximum mass of compact stellar objects, thus leading to a modifications of the "classical" Buchdahl limit [10].

The maximum allowable mass - radius ratio in the case of stable charged compact general relativistic objects was obtained in [11], by generalizing to the charged case the methods used for neutral stars by Buchdahl [9] and Straumann [12].

On the other hand, we cannot exclude *a priori* the possibility that the cosmological constant, as a manifestation of vacuum energy, may play an important role not only at galactic or cosmological scales, but also at the level of elementary particles (the very successful phenomenological bag model of hadrons requires the existence of the vacuum energy inside and outside strongly interacting particles). With the use of the generalized Buchdahl identity [10], it can be rigorously proven that the existence of a non-negative Λ imposes a lower bound on the mass M and density ρ of general relativistic objects of radius R, which is given by [13]

$$2M \geq \frac{8\pi\Lambda}{6}R^3, \qquad \rho = \frac{3M}{4\pi R^3} \geq \frac{\Lambda}{2} =: \rho_{\min}. \tag{1}$$

Therefore, the existence of the cosmological constant implies the existence of an absolute minimum mass and density in the universe. No object present in relativity can have a density that is smaller than $\rho_{\min}$. For $\Lambda > 0$ this result also implies a minimum density for stable fluctuations in energy density.

It is the purpose of the present paper to consider the problem of the existence of a minimum mass-radius ratio for compact electrically charged general relativistic objects. We rigorously prove that a lower bound for the ratio M/R does exist for charged objects with non-zero electric charge Q. This result follows from the same Buchdahl type inequality which has been extensively used for the proof of the existence of an upper bound for the mass-radius ratio. From the study of the behavior of the Ricci scalar invariants we obtain several bounds on the total charge, mass and the vacuum energy for compact charged objects. For the electrically charged objects with the minimum mass-radius ratio we analyze their stability problem by considering the total energy (including the gravitational one), and requiring for the particles to be in a state of minimum energy. This leads to a representation of the mass and radius of the charged objects with minimum mass-radius ratio in terms of the charge and vacuum energy only.

The present paper is organized as follows. The generalized Buchdahl inequality for charged objects in the presence of a vacuum energy (a cosmological constant) is derived in Section II. In Section III we obtain some bounds on the total charge and mass of compact charged objects from the study of the Ricci scalar invariants. The total energy (including the gravitational one) and the stability of the objects with minimum mass-radius ratio is considered in Section IV. We discuss and conclude our results in Section V.

Throughout this paper we use the Landau-Lifshitz conventions [14] for the metric signature $(+,-,-,-)$ and for the field equations, and a system of units with $c = G = \hbar = 1$.

II. GENERALIZED BUCHDAHL INEQUALITY FOR CHARGED OBJECTS

For a static general relativistic spherically symmetric configuration the interior line element is given by

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2}$$

The properties of a charged compact general relativistic object can be completely described by the structure equations, which are given by

$$\frac{dm}{dr} = 4\pi\rho r^2 + \frac{Q}{r}\frac{dQ}{dr}, \tag{3}$$

$$\frac{dp}{dr} = -\frac{(\rho+p)\left[m + 4\pi r^3\left(p - \frac{2B}{3}\right) - \frac{Q^2}{r}\right]}{r^2\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{3}Br^2\right)} + \frac{Q}{4\pi r^4}\frac{dQ}{dr}, \tag{4}$$

$$\frac{d\nu}{dr} = \frac{2\left[m + 4\pi r^3\left(p - \frac{2B}{3}\right) - \frac{Q^2}{r}\right]}{r^2\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{8\pi}{3}Br^2\right)}, \tag{5}$$

where $\rho(r)$ is the energy density of the matter, $p(r)$ is the thermodynamic pressure, $m(r)$ is the mass and

$$Q(r) = 4\pi\int_0^r e^{\frac{\nu+\lambda}{2}} r^2 j^0 dr, \tag{6}$$

is the electric charge inside radius r, respectively. The electric current inside the charged object is given by $j^\mu = (j^0, 0, 0, 0)$. By analogy with the bag model of hadrons we also assume the presence of an effective constant vacuum energy density B (a cosmological constant) inside and outside the charged object. Eqs. (3)-(5) represent the generalization of the structure equations for general relativistic static charged objects, introduced for the first time in [15], by taking into account the existence of a non-zero vacuum energy.

Generally p and ρ are related by an equation of state of the form $p = p(\rho)$. The structure equations Eqs. (3)-(5) must be considered together with the boundary conditions $p(R) = 0$, $p(0) = p_c$, $\rho(0) = \rho_c$ and $Q(0) = 0$, where ρ_c, p_c are the central density and pressure, respectively.

With the use of Eqs. (3)-(5) it is easy to show that the function $\zeta = \exp(\nu/2) > 0$, $\forall r \in [0, R]$, obeys the equation

$$\sqrt{1-\frac{2m}{r}+\frac{Q^2}{r^2}-\frac{8\pi}{3}Br^2}\frac{1}{r}\frac{d}{dr}\left[\sqrt{1-\frac{2m}{r}+\frac{Q^2}{r^2}-\frac{8\pi}{3}Br^2}\frac{1}{r}\frac{d\zeta}{dr}\right]=\frac{\zeta}{r}\left[\frac{d}{dr}\frac{m}{r^3}+\frac{Q^2}{r^5}\right]. \quad (7)$$

For $Q = 0$ and $B = 0$ we obtain the equation considered in [12]. Since the density ρ does not increase with increasing r, the mean density of the matter $<\rho>= 3m(r)/4\pi r^3$ inside radius r does not increase either. Therefore we assume that inside a compact general relativistic object the condition

$$\frac{d}{dr}\frac{m}{r^3} < 0, \quad (8)$$

holds independently of the equation of state of dense matter and of the electric charge distribution inside the object.

By defining a new function

$$\eta(r) = \int_0^r \frac{r'}{\sqrt{1-\frac{2m(r')}{r'}+\frac{Q^2(r')}{r'^2}-\frac{8\pi}{3}Br'^2}}\left[\int_0^{r'}\frac{Q^2(r'')\zeta(r'')}{r''^5\sqrt{1-\frac{2m(r'')}{r''}+\frac{Q^2(r'')}{r''^2}-\frac{8\pi}{3}Br''^2}}dr''\right]dr', \quad (9)$$

denoting $\Psi = \zeta - \eta$, and introducing a new independent variable $\xi(r)$ by means of the transformation [11], [12],

$$\xi(r) = \int_0^r r'\left[1-\frac{2m(r')}{r'}+\frac{Q^2(r')}{r'^2}-\frac{8\pi}{3}Br'^2\right]^{-\frac{1}{2}}dr', \quad (10)$$

from Eq. (9) we obtain the basic result that inside all stable stellar type charged general relativistic matter distributions the condition

$$\frac{d^2\Psi}{d\xi^2} < 0, \quad (11)$$

must hold for all $r \in [0, R]$. Using the mean value theorem [12] we conclude that

$$\frac{d\Psi}{d\xi} \leq \frac{\Psi(\xi) - \Psi(0)}{\xi}, \tag{12}$$

or, taking into account that $\Psi(0) > 0$ it follows that,

$$\Psi^{-1}\frac{d\Psi}{d\xi} \leq \frac{1}{\xi}. \tag{13}$$

In the following we denote

$$\alpha(r) = 1 - \frac{Q^2(r)}{2m(r)r} + \frac{4\pi}{3}B\frac{r^3}{m(r)}. \tag{14}$$

In the initial variables the inequality (13) takes the form

$$\frac{1}{r}\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\left[\frac{1}{2}\frac{d\nu}{dr}e^{\frac{\nu(r)}{2}} - \frac{r}{\sqrt{1-\frac{2\alpha(r)m(r)}{r}}}\int_0^r \frac{Q^2(r')e^{\frac{\nu(r')}{2}}}{r'^5\sqrt{1-\frac{2\alpha(r')m(r')}{r'}}}dr'\right] \leq$$
$$\frac{e^{\frac{\nu(r)}{2}} - \int_0^r r'\left[1-\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}\left\{\int_0^{r'}\left[1-\frac{2\alpha(r'')m(r'')}{r''}\right]^{-\frac{1}{2}}\frac{Q^2(r'')e^{\frac{\nu(r'')}{2}}}{r''^5}dr''\right\}dr'}{\int_0^r r'\left[1-\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}dr'}. \tag{15}$$

For any stable charged compact objects m/r^3 does not increase outwards. We suppose that for all $r' \leq r$ we have

$$\frac{\alpha(r')m(r')}{r'} \geq \frac{\alpha(r)m(r)}{r}\left(\frac{r'}{r}\right)^2, \tag{16}$$

or, equivalently,

$$\frac{2m(r')}{r'} - \frac{2m(r)}{r}\left(\frac{r'}{r}\right)^2 \geq \frac{Q^2(r')}{r'^2} - \frac{Q^2(r)}{r^2}\left(\frac{r'}{r}\right)^2. \tag{17}$$

We assume that inside the compact stellar object the charge $Q(r)$ satisfies the general condition

$$\frac{Q^2(r'')e^{\frac{\nu(r'')}{2}}}{r''^5} \geq \frac{Q^2(r')e^{\frac{\nu(r')}{2}}}{r'^5} \geq \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}, r'' \leq r' \leq r. \tag{18}$$

Therefore we can evaluate the terms in Eq. (15) as follows. For the term in the denominator of the right hand side of Eq. (15) we obtain:

$$\left\{\int_0^r r'\left[1-\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}dr'\right\}^{-1} \leq \frac{2\alpha(r)m(r)}{r^3}\left[1-\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\right]^{-1}. \tag{19}$$

For the second term in the bracket of the left hand side of Eq. (15) we have

$$\int_0^r \left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}\frac{Q^2\left(r'\right)e^{\frac{\nu\left(r'\right)}{2}}}{r'^5}dr' \geq$$
$$\frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\int_0^r\left[1-\frac{2\alpha\left(r\right)m(r)}{r}\left(\frac{r'}{r}\right)^2\right]^{-\frac{1}{2}}dr' =$$
$$\frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\left[\frac{2\alpha(r)m(r)}{r^3}\right]^{-\frac{1}{2}}\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r}}\right]. \tag{20}$$

The second term in the nominator of the right hand side of Eq. (15) can be evaluated as

$$\int_0^r r'\left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}\left\{\int_0^{r'}\left[1-\frac{2\alpha\left(r''\right)m\left(r''\right)}{r''}\right]^{-\frac{1}{2}}\frac{Q^2\left(r''\right)e^{\frac{\nu\left(r''\right)}{2}}}{r''^5}dr''\right\}dr' \geq$$
$$\int_0^r r'\left[1-\frac{2\alpha\left(r'\right)m\left(r'\right)}{r'}\right]^{-\frac{1}{2}}\frac{Q^2\left(r'\right)e^{\frac{\nu\left(r'\right)}{2}}}{r'^4}\left[\frac{2\alpha(r')m(r')}{r'}\right]^{-\frac{1}{2}}\arcsin\left[\sqrt{\frac{2\alpha(r')m(r')}{r'}}\right]dr' \geq$$
$$\frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\int_0^r r'^2\left[1-\frac{2\alpha(r)m(r)}{r^3}r'^2\right]^{-\frac{1}{2}}\left[\frac{2\alpha(r)m(r)}{r^3}r'^2\right]^{-\frac{1}{2}}\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r^3}}r'\right]dr' =$$
$$\frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^2}\left[\frac{2\alpha(r)m(r)}{r}\right]^{-\frac{3}{2}}\left\{\sqrt{\frac{2\alpha(r)m(r)}{r}}-\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r}}\right]\right\} \tag{21}$$

In order to obtain the inequality (21) we have also used the property of monotonic increase in the interval $x \in [0,1]$ of the function $\arcsin x/x$.

Using Eqs. (19)-(21), Eq. (15) becomes:

$$\left[1-\sqrt{1-\frac{2\alpha(r)m(r)}{r}}\right]\frac{m(r)+4\pi r^3\left(p-\frac{2}{3}B\right)-\frac{Q^2}{r}}{r^3\sqrt{1-\frac{2\alpha(r)m(r)}{r}}} \leq \frac{2\alpha(r)m(r)}{r^3}+\frac{Q^2}{r^4}\left\{\frac{\arcsin\left[\sqrt{\frac{2\alpha(r)m(r)}{r}}\right]}{\sqrt{\frac{2\alpha(r)m(r)}{r}}}-1\right\}. \tag{22}$$

The Buchdahl type inequality given by Eq. (22) is valid for all r inside the electrically charged object. It naturally leads to the existence of a maximum mass-radius ratio for general relativistic objects.

Consider first the neutral case $Q = 0$ and assume that the vacuum energy is zero, $B = 0$. We assume that at the surface of the compact object, defined by a radius $r = R$, the thermodynamical pressure p vanishes, $p(R) = 0$. By evaluating (22) for $r = R$ we obtain $(1-2M/R)^{-1/2} \leq 2\left[1-(1-2M/R)^{-1/2}\right]^{-1}$, leading to the well-known result $2M/R \leq 8/9$ [9, 12]. The maximum mass-radius ratio for charged object, representing the generalization to the charged case of the Buchdahl limit, was considered, and extensively discussed in the case of a vanishing vacuum energy $B = 0$, in [11].

III. MINIMUM MASS-RADIUS RATIO FOR CHARGED GENERAL RELATIVISTIC OBJECTS

Eq. (22) also implies the existence of a minimum mass-radius ratio for compact charged general relativistic objects. This can be shown as follows. For small values of the argument the function $\arcsin x/x - 1$ can be approximated as $\arcsin x/x - 1 \approx x^2/6$. Therefore, at the vacuum boundary $r = R$ of the charged object, Eq. (22) can be written in an equivalent form as

$$\sqrt{1-\frac{2M}{R}+\frac{Q^2}{R^2}-\frac{8\pi}{3}BR^2} \geq \frac{M-\frac{Q^2}{R}-\frac{8\pi}{3}BR^3}{3M-2\frac{Q^2}{R}+\frac{Q^2}{6R^2}\left(2M-\frac{Q^2}{R}+\frac{8\pi}{3}BR^3\right)}. \tag{23}$$

By introducing a new variable u defined as

$$u = \frac{M}{R} - \frac{Q^2}{2R^2} + \frac{4\pi}{3}BR^2, \tag{24}$$

Eq. (23) takes the form

$$\sqrt{1-2u} \geq \frac{u-a}{bu-a}, \tag{25}$$

where we denoted

$$a = \frac{Q^2}{2R^2} + 4\pi BR^2, \tag{26}$$

and

$$b = 3 + \frac{Q^2}{3R^2}, \tag{27}$$

respectively. Then, by squaring, we can reformulate the condition given by Eq. (25) as

$$u\left[2b^2u^2 - \left(b^2+4ab-1\right)u + 2a\left(a+b-1\right)\right] \leq 0, \tag{28}$$

or, equivalently,

$$u\left(u-u_1\right)\left(u-u_2\right) \leq 0, \tag{29}$$

where

$$u_1 = \frac{b^2+4ab-1-(1-b)\sqrt{(1+b)^2-8ab}}{4b^2}, \tag{30}$$

and

$$u_2 = \frac{b^2 + 4ab - 1 + (1-b)\sqrt{(1+b)^2 - 8ab}}{4b^2}, \tag{31}$$

respectively.

Since $u \geq 0$, Eq. (29) is satisfied if $u \leq u_1$ and $u \geq u_2$, or $u \geq u_1$ and $u \leq u_2$. However, the condition $u \geq u_1$ contradicts the upper bound which follows from Eq. (24), and which has been discussed in detail in [11]. Therefore, Eq. (29) is satisfied if and only if for all values of the physical parameters the condition $u \geq u_2$ holds. This is equivalent to the existence of a minimum bound for the mass-radius ratio of compact anisotropic objects, which is given by

$$u \geq \frac{2a}{1+b}, \tag{32}$$

where we have taken into account that $(1+b)^2 >> 8ab$. Using the expressions of a, b and u as defined above yields the minimum mass-radius ratio for electrically charged general relativistic objects as

$$\frac{2M}{R} \geq \frac{3}{2}\frac{Q^2}{R^2}\frac{1 + \frac{8\pi}{9}B\frac{R^4}{Q^2} - \frac{4\pi}{27}BR^2 + \frac{Q^2}{18R^2}}{1 + \frac{Q^2}{12R^2}}. \tag{33}$$

Let us neglect the vacuum energy component ($B = 0$) for the moment. Then the minimum mass-radius ratio given by Eq. (33) takes the following form

$$\frac{2M}{R} \geq \frac{3}{2}\frac{Q^2}{R^2}\frac{1 + \frac{Q^2}{18R^2}}{1 + \frac{Q^2}{12R^2}}. \tag{34}$$

Eq. (34) can be Taylor expanded in the term Q^2/R^2. The assumption $Q^2/R^2 \ll 1$ is natural since the total charge is always many orders smaller than the radii of the charged general relativistic objects. Therefore we find

$$\frac{2M}{R} \geq \frac{3}{2}\frac{Q^2}{R^2}\left[1 - \frac{Q^2}{36R^2} + O(Q^2/R^2)^4\right], \tag{35}$$

In the lowest order in Q^2/R^2 the mass-radius ratio is bounded from below by $2M/R \geq 3Q^2/2R^2$. For $Q = 0$ and $B \neq 0$ we obtain the minimum mass for neutral objects in the presence of the vacuum energy B, given by Eq. (1).

If in Eq. (33) we neglect the term containing the product BQ^2 and again assume that $Q^2/R^2 \ll 1$, the minimum mass of a charged particle can be generally represented in an approximate form as

$$M \geq \frac{2\pi}{3}BR^3 + \frac{3}{4}\frac{Q^2}{R}. \tag{36}$$

Furthermore, by expressing the mass of a spherically symmetric object in terms of the mean density $< \rho >= 3M/4\pi R^3$, Eq. (36) gives the following lower bound on the mean density of a charged general relativistic object

$$\langle\rho\rangle \geq \frac{B}{2} + \frac{9}{16\pi}\frac{Q^2}{R^4}. \tag{37}$$

It should be noted that in the absence of charge, the lower bound in Eq. (37) only depends on the vacuum energy component B and is independent of the object's radius R. Hence, the bound due to the vacuum energy must be regarded as an absolute bound, valid on all scales of interest. On the other hand, the additional contribution to the minimal density due to the presence of the charge depends on the radius.

For large astrophysical objects, the additional charge term is suppressed by four orders of magnitude in the radius. Therefore, the charge term can only have an important effect if relatively small objects with high electric charge are considered. To further elucidate this point, let us introduce the surface charge density given by

$$\sigma = \frac{Q}{4\pi R^2}, \tag{38}$$

where it should be noted that the charge term Q takes into account the total charge of the compact object. Using this definition, Eq. (37) leads to

$$\langle\rho\rangle \geq \frac{B}{2} + 9\pi\sigma^2. \tag{39}$$

It is now obvious that the charge can have a significant effect on the allowed mean density of the compact charged general relativistic objects. In particular, configurations where the charge is mainly located near the surface of the object yield a strong lower bound on the mean density of those general relativistic objects.

IV. MASS-RADIUS RATIO CONSTRAINTS FROM THE RICCI INVARIANTS

In order to find a general restriction for the total charge Q a compact stable object can acquire in the presence of a cosmological constant we consider the behavior of the Ricci invariants

$$r_0 = R_i^i = R, \tag{40}$$

$$r_1 = R_{ij}R^{ij}, \tag{41}$$

and

$$r_2 = R_{ijkl}R^{ijkl}, \tag{42}$$

respectively.

If the general static line element is regular, satisfying the conditions $e^{\nu(0)}$ =constant$\neq 0$ and $e^{\lambda(0)} = 1$, then the Ricci invariants are also non-singular functions throughout the compact object. In particular for a regular space-time the invariants are non-vanishing at the origin $r = 0$. For the invariant r_2 we find

$$\begin{aligned} r_2 = & \left[8\pi\left(\rho + p\right) - \frac{4m}{r^3} - \frac{16\pi}{3}B + \frac{6Q^2}{r^4}\right]^2 + 2\left(8\pi p + \frac{2m}{r^3} - \frac{16\pi}{3}B - \frac{2Q^2}{r^4}\right)^2 + \\ & 2\left(8\pi\rho - \frac{2m}{r^3} + \frac{16\pi}{3}B + \frac{2Q^2}{r^4}\right)^2 + 4\left(\frac{2m}{r^3} + \frac{8\pi}{3}B - \frac{Q^2}{r^4}\right)^2. \end{aligned} \tag{43}$$

For a monotonically decreasing interior electric field $Q^2/8\pi r^4$, the function r_2 is regular and monotonically decreasing throughout the star. Therefore it satisfies the condition $r_2(R) < r_2(0)$, leading to the following equation quadratic in Q^2/R^4

$$\left(\frac{Q^2}{R^4}\right)^2 + \left(\frac{Q^2}{R^4}\right)\frac{16\pi}{7}B - \frac{24}{7}\pi^2 p_c^2 - \frac{16}{7}\pi^2 p_c\rho_c - \frac{40}{21}\pi^2\rho_c^2 + \frac{32}{21}\pi^2\langle\rho\rangle^2 + \frac{32}{7}\pi^2 p_c B - \frac{32}{21}\pi^2\rho_c B < 0, \tag{44}$$

where we assumed that at the surface of the star the matter density vanishes, $\rho(R) = 0$. We rewrite this equation in the form

$$\left(\frac{Q^2}{R^4} - q_+\right)\left(\frac{Q^2}{R^4} - q_-\right) < 0, \tag{45}$$

where the two roots are given by

$$q_\pm = -\frac{24\pi B}{21} \pm \frac{2\pi\rho_c\sqrt{6}}{21}\sqrt{35 + 42\frac{p_c}{\rho_c}\left(1 - \frac{2B}{\rho_c}\right) + 63\frac{p_c^2}{\rho_c^2} - 28\frac{\langle\rho\rangle^2}{\rho_c^2} + 28\frac{B}{\rho_c} + 24\frac{B^2}{\rho_c^2}}. \tag{46}$$

Since the term Q^2/R^4 is positive definite, Eq. (45) can only be satisfied if

$$q_- < \frac{Q^2}{R^4} \quad \text{and} \quad q_+ > \frac{Q^2}{R^4}. \tag{47}$$

This first condition is simply the positivity of Q^2/R^4, whereas the second condition yields the upper bound

$$\frac{Q^2}{R^4} < \frac{2\pi\rho_c\sqrt{6}}{21}\sqrt{35 + 42\frac{p_c}{\rho_c}\left(1 - \frac{2B}{\rho_c}\right) + 63\frac{p_c^2}{\rho_c^2} - 28\frac{\langle\rho\rangle^2}{\rho_c^2} + 28\frac{B}{\rho_c} + 24\frac{B^2}{\rho_c^2}} - \frac{24\pi B}{21}, \tag{48}$$

which for vanishing dark energy simplifies to

$$\frac{Q^2}{R^4} < \frac{2\pi\rho_c\sqrt{6}}{21}\sqrt{35 + 42\frac{p_c}{\rho_c} + 63\frac{p_c^2}{\rho_c^2} - 28\frac{\langle\rho\rangle^2}{\rho_c^2}}. \tag{49}$$

Another condition on $Q(R)$ can be obtained from the study of the scalar

$$r_1 = \left(8\pi\rho + 8\pi B + \frac{Q^2}{r^4}\right)^2 + 3\left(8\pi p - 8\pi B - \frac{Q^2}{r^4}\right)^2 + \frac{64\pi p Q^2}{r^4} - \frac{64\pi B Q^2}{r^4}. \tag{50}$$

Under the same assumptions of regularity and monotonicity for the function r_1 and considering that the surface density is vanishing we obtain for the surface value of the monotonically decreasing electric field the upper bound

$$\frac{Q^2}{R^4} < 4\pi\left(\rho_c + B\right)\sqrt{1 + 3\left(\frac{p_c - B}{\rho_c + B}\right)^2 - \frac{4B^2}{\left(\rho_c + B\right)^2}}. \tag{51}$$

The invariant r_0 leads to the trace condition $\rho_c + B > 3p_c - 3B$ of the energy-momentum tensor that holds at the center of the fluid spheres.

V. TOTAL ENERGY AND STABILITY OF CHARGED OBJECTS WITH MINIMUM MASS-RADIUS RATIO

As another application of the obtained minimum mass -radius ratio for charged objects we derive an explicit expression for the total energy of compact charged general relativistic objects with minimum mass-radius ratio.

The total energy E (including the gravitational field contribution) inside an equipotential surface S of radius R can be defined, according to [16], to be

$$E = E_M + E_F = \frac{1}{8\pi}\xi_s\int_S [K] dS, \tag{52}$$

where ξ^i is a Killing vector field of time translation, ξ_s its value at S and $[K]$ is the jump across the shell of the trace of the extrinsic curvature of S, considered as embedded in the 2-space $t =$ constant. $E_M = \int_S T_i^k \xi^i \sqrt{-g} dS_k$ and E_F are the energy of the matter and of the gravitational field, respectively, with T_i^k the energy-momentum tensor of the matter. This definition is manifestly coordinate invariant.

For a static charged spherically symmetric system in the presence of a cosmological constant the total energy inside the radius R is

$$E = R\left[1 - \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} - \frac{8\pi}{3}BR^2\right)^{1/2}\right]\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} - \frac{8\pi}{3}BR^2\right)^{1/2}. \tag{53}$$

For the minimum mass-radius ratio charged object, with $2M/R = (3/2)\,Q^2/R^2 + 4\pi BR^2/3$, the total energy can be expressed in terms of the radius, charge and vacuum energy only as

$$E = R\left[1-\left(1-\frac{Q^2}{2R^2}-4\pi BR^2\right)^{1/2}\right]\left(1-\frac{Q^2}{2R^2}-4\pi BR^2\right)^{1/2}. \tag{54}$$

For a stable configuration, the energy should have a minimum,

$$\frac{\partial E}{\partial R} = 0, \tag{55}$$

a condition which gives the following algebraic equation determining R as a function of B and Q:

$$1+\frac{Q^2}{2R^2}-12\pi BR^2-\frac{1-8\pi BR^2}{\sqrt{1-\frac{Q^2}{2R^2}-4\pi BR^2}} = 0. \tag{56}$$

By Taylor-expanding the square root and keeping only the first order terms in Q^2 and B we obtain the radius of the stable minimum mass charged configuration as

$$R = (24\pi)^{-1/4}\frac{\sqrt{Q}}{B^{1/4}}. \tag{57}$$

Therefore the minimum mass of a charged object can be expressed as a function of the vacuum energy density B and the electric charge in the form

$$M = \frac{7}{9}(24\pi)^{1/4}Q^{3/2}B^{1/4}. \tag{58}$$

By eliminating the vacuum energy between Eqs. (57) and (58) we obtain the following mass-radius-charge relation:

$$M = \frac{7}{9}\frac{Q^2}{R}. \tag{59}$$

The surface charge density of the stable objects with minimum mass-radius ratio can be expressed in terms of the vacuum energy only as

$$\sigma = \sqrt{\frac{3B}{2\pi}}. \tag{60}$$

VI. DISCUSSIONS AND FINAL REMARKS

In the present paper we have shown that a minimum mass-radius ratio for charged stable compact general relativistic objects do exist, and it is the direct consequence of the same

Buchdahl inequality giving the upper bound for the mass-radius ratio. In the case of the minimum mass-radius ratio it is also possible to obtain explicit inequalities giving the lower bound for $2M/R$ as an explicit function of the charge Q and of the vacuum energy density B. The condition of the thermodynamic stability of the minimum mass object leads to an explicit representation of the mass and radius in terms of the charge Q and of the vacuum energy B only.

A very interesting and long debated question is the possible applicability of general relativity to describe elementary particles, and, in particular, the electron. In 1919 Einstein [17] suggested a modification of the geometrical terms of the gravitational field equations of general relativity with only the energy-momentum tensor of the electromagnetic field being present in place of the energy-momentum tensor of matter. In this theory the self-stabilizing stresses are of non-electromagnetic origin, the gravitational forces providing the necessary stability of the electron and also contributing to its mass. However, the breaking of the vacuum energy inside and outside charged particles may provide an alternative mechanism for the stabilization of the charged elementary particles.

With respect to the scaling of the parameters B and Q of the form $B \to kB$ and $Q \to lQ$, the minimum mass and radius have the following scaling behaviors:

$$R \to l^{3/2}k^{-1/4}R, M \to l^{3/2}k^{1/4}M. \tag{61}$$

For a constant charge $l = 1$, and therefore charged particles with different masses can be constructed for different values of the vacuum energy by starting from a minimum mass configuration.

In the case of an electron, with mass $m_e = 0.51$ MeV and charge $e = \alpha^{1/2} = 137^{-1/2}$, where α is the fine structure constant, from Eq. (58) it follows that the value of the vacuum energy B_e necessary to stabilize the configuration is $B_e^{1/4} = 8.91$ MeV. In the case of quarks and hadrons, the value of the vacuum energy (bag constant) necessary to stabilize the bag is $B_{QCD}^{1/4} = 145$ MeV [3]. On the other hand, the radius of the electron obtained with the use of $B_e^{1/4} = 8.91$, given by Eq. (57), is $R_e = 0.011$ MeV^{-1} $= 2.19$ fm (1 Mev$= 5.064 \times 10^{-3}$ fm^{-1}). Therefore Eqs. (57) and (58) can give a satisfactory description of the basic classical physical parameters of the electron.

By interpreting the charge Q in Eq. (58) as a generalized charge, we can apply it even for strongly interacting particles. In the case of strong interactions, the strong coupling constant

α_s is a function of the particle momenta. The quark-quark-gluon coupling constant for the simplest hadrons is $\alpha_s \approx 0.12$, and, by defining the generalized charge as $Q_{QCD} \approx \alpha_s^{1/2}$, with the use of the value of the bag constant as obtained in quantum chromodynamics, we obtain for the mass of the quarks a reasonable value of the order of $m_q = 67.75$ MeV.

The possibility that general relativity or a similar geometric description may play an important role at the scale of elementary particles is still very controversial. On the other hand, the possibility of the estimation of the mass of the charged elementary particles from general relativistic considerations in the framework of a broken vacuum model can perhaps give a better understanding of the deep connection between micro- and macro-physics.

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