ON THE MONOTONICITY AND LOG-CONVEXITY OF THE SECOND KIND ONE-PARAMETER HOMOGENEOUS FUNCTION

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ABSTRACT. That $\mathcal{H}_{2f}(p;a,b) := \mathcal{H}_f(p,1-p;a,b)$ is called a second kind one-parameter homogeneous function, of which monotonicity and log-convexity depend on the sign of $J=(x-y)(xI)_x$, where $I=(\ln f)_{xy}$. By straightforward computations, some conclusions on the monotonicity and log-convexity of $\mathcal{H}_{2f}(p)$ are presented, where $f(x,y)=L(x,y)$, $A(x,y)$, $E(x,y)$ and $D(x,y)$, from which other new refined inequalities involving logarithmic mean, exponential mean, power-exponential mean and power mean etc. are derived.

1. Introduction

The extended mean and Gini mean were generalized to the so-called two-parameter homogeneous functions by the author, which is defined by [20]

Definition 1. *Assume $f\colon \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+\cup\{0\}$ is an n-order homogeneous function of variables x and y which is continuous and exists first partial derivatives, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ with $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$.*

If $f(x,y)>0$ for $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ with $x\neq y$ and $f(x,x)=0$ for all $x\in\mathbb{R}_+$, then define that

$$\mathcal{H}_f(p,q;a,b) = \left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}} (p\neq q, pq\neq 0), \tag{1.1}$$

$$\mathcal{H}_f(p,p;a,b) = \lim_{q\to p}\mathcal{H}_f(p,q;a,b) = G_{f,p}(p=q\neq 0), \tag{1.2}$$

where

$$G_{f,p} = G_f^{\frac{1}{p}}(a^p,b^p),\quad G_f(x,y)=\exp\left(\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right), \tag{1.3}$$

$f_x(x,y)$ and $f_y(x,y)$ denote partial derivative with respect to first and second variable of $f(x,y)$ respectively.

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If $f(x,y)>0$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ then define further

$$(1.4)\qquad \mathcal{H}_f(p,0;a,b) = \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{\frac{1}{p}} (p\neq 0, q=0),$$

$$(1.5)\qquad \mathcal{H}_f(0,q;a,b) = \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{\frac{1}{q}} (p=0, q\neq 0),$$

$$(1.6)\qquad \mathcal{H}_f(0,0;a,b) = \lim_{p\to 0}\mathcal{H}_f(p,0;a,b) = a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}} (p=q=0).$$

Since $f(x,y)$ is a homogeneous function, $\mathcal{H}_f(p,q;a,b)$ is also one and called a homogeneous function with parameters p and q, and simply denote by $\mathcal{H}_f(p,q)$.

Let $q=p+1$ in the Definition 1. Then $\mathcal{H}_f(p,p+1;a,b)$ is called a one-parameter homogeneous function and denote by $\mathcal{H}_{1f}(p,;a,b)$ [21], which contain one-parameter mean and Lehmer mean (see [3, 18]).

The author investigated the monotonicity and log-convexity of two-parameter and one-parameter homogeneous functions, and obtained a series of interesting and useful results (see [20, 21, 22, 23, 24]).

The *one-parameter mean* is relative to the *two-parameter mean*. In general, $\mathcal{H}_f(p,q)$ may be called a one-parameter homogeneous function provided there exists a sort of given relations between its parameters p and q. The problem is which of one-parameter functions could be investigated and worth investigating.

Let $q=1-p$ in the Definition 1. Then we can obtain another one-parameter homogeneous function $\mathcal{H}_f(p,1-p;a,b)$. For avoiding confusion, we call $\mathcal{H}_f(p,p+1;a,b)=\mathcal{H}_{1f}(p,;a,b)$ first kind one-parameter homogeneous function, while $\mathcal{H}_f(p,1-p;a,b)$ second kind one and denote by $\mathcal{H}_{2f}(p;a,b)$ or $\mathcal{H}_{2f}(p)$.

Using log-convexity of two-parameter or first kind one-parameter homogeneous functions, the author obtained results involving the monotonicity of the second kind homogeneous functions, which is read as follows [21, 24]:

Theorem 1 ([21, Corollary 1] or [24, Corollary 3])**.** *Let $f(x,y)$ be a positive symmetric n-order homogenous function defined on $\mathbb{R}_+\times\mathbb{R}_+\backslash\{(x,x):x\in\mathbb{R}_+\}$ which is three-time differentiable. If $J=(x-y)(xI)_x<(>)0$ where $I=(\ln f)_{xy}$, then*

1) $\mathcal{H}_{2f}(p)$ is strictly decreasing (increasing) in $p\in(0,\frac{1}{2})$ and increasing (decreasing) in $p\in(\frac{1}{2},1)$.

2) If $f(x,y)$ is defined on $\mathbb{R}_+\times\mathbb{R}_+$ and symmetric with respect to x and y further, then $\mathcal{H}_{2f}(p)$ is strictly decreasing (increasing) in $p\in(-\infty,\frac{1}{2})$ and increasing (decreasing) in $p\in(\frac{1}{2},+\infty)$.

The aim of this paper is to give a new proof of 1 and investigate the log-convexity in parameter p of second kind homogeneous function. As applications, some new inequalities will be presented.

2. Basic Concepts and Main Results

we give the definition of one-parameter homogeneous functions firstly.

Definition 2. *Assume $f\colon\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+\cup\{0\}$ is an n-order homogeneous function of variables x and y which is continuous and exist first partial derivatives, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ with $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$.*

If $f(x,y)>0$ for $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ with $x\neq y$ and $f(x,x)=0$ for all $x\in\mathbb{R}_+$, then define that

$$\mathcal{H}_{2f}(p;a,b) = \left(\frac{f(a^p,b^p)}{f(a^{1-p},b^{1-p})}\right)^{\frac{1}{2p-1}} \quad \text{if } p(1-p)(p-\frac{1}{2})\neq 0, \tag{2.1}$$

$$\mathcal{H}_{2f}(\frac{1}{2};a,b) = \lim_{p\to\frac{1}{2}}\mathcal{H}_{2f}(p;a,b) = G_{f,\frac{1}{2}}, \tag{2.2}$$

where $G_{f,p}$ is defined by (1.3).

If $f(x,y)>0$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ then define further

$$\mathcal{H}_{2f}(0;a,b) = \mathcal{H}_{2f}(1;a,b) = \frac{f(a,b)}{f(1,1)}. \tag{2.3}$$

Since $f(x,y)$ is a homogeneous function, $\mathcal{H}_{2f}(p;a,b)$ is also one and called a second kind homogeneous function with a parameter p and simply called second kind one-parameter homogeneous function.

Example 1. *From Definition 2, put $f(x,y)=L(x,y)=(x-y)/\ln(x/y)(x,y>0,x\neq y)$, $f(x,x)=L(x,x)=x$, then*

$$\mathcal{H}_{2L}(p;a,b) = \begin{cases} \left(\frac{(1-p)(a^p-b^p)}{p(a^{1-p}-b^{1-p})}\right)^{\frac{1}{2p-1}}, & p\neq\frac{1}{2}; \\ E^2(\sqrt{a},\sqrt{b}), & p=\frac{1}{2}, \end{cases} \tag{2.4}$$

where $E(x,y)=e^{-1}(x^x/y^y)^{1/(x-y)}(x\neq y)$, $E(x,x)=x$.

Example 2. *Put $f(x,y)=A(x,y)=\frac{1}{2}(x+y)(x,y>0)$, then*

$$\mathcal{H}_{2A}(p;a,b) = \begin{cases} \left(\frac{a^p+b^p}{a^{1-p}+b^{1-p}}\right)^{\frac{1}{2p-1}}, & p\neq\frac{1}{2}; \\ Z^2(\sqrt{a},\sqrt{b}), & p=\frac{1}{2}, \end{cases} \tag{2.5}$$

where $Z(x,y)=x^{x/(x+y)}y^{y/(x+y)}$.

Example 3. *Put $f(x,y)=E(x,y)=e^{-1}(x^x/y^y)^{1/(x-y)}(x,y>0,x\neq y)$, $f(x,x)=E(x,x)=x$, then*

$$\mathcal{H}_{2E}(p;a,b) = \begin{cases} \left(\frac{E(a^p,b^p)}{E(a^{1-p},b^{1-p})}\right)^{\frac{1}{2p-1}}, & p\neq\frac{1}{2}; \\ Y^2(\sqrt{a},\sqrt{b}), & p=\frac{1}{2}, \end{cases} \tag{2.6}$$

where $Y(x,y)=E(x,y)\exp\left(1-G^2(x,y)/L^2(x,y)\right)(,x\neq y)$, $Y(x,x)=x$.

Example 4. *Put $f(x,y)=D(x,y)=|x-y|(x,y>0,x\neq y)$, then*

$$\mathcal{H}_{2D}(p;a,b) = \begin{cases} |\frac{a^p-b^p}{a^{1-p}-b^{1-p}}|^{\frac{1}{2p-1}}, & p\neq 0,1,\frac{1}{2}; \\ e^2E^2(\sqrt{a},\sqrt{b}), & p=\frac{1}{2}, \end{cases} \tag{2.7}$$

$\mathcal{H}_{2L}(x,y;p)$ is called second kind one-parameter logarithmic mean. In the same way, $\mathcal{H}_{2A}(x,y;p)$ and $\mathcal{H}_{2E}(x,y;p)$ are called second kind one-parameter arithmetic mean and exponential mean, respectively.

Since $D(x,y)$ is no a certain mean of positive numbers x and y, but a difference function, we call $\mathcal{H}_{2D}(x,y;p)$ second kind one-parameter homogeneous difference function.

Concerning the log-convexity of the second kind one-parameter homogeneous functions, we have the following results.

Theorem 2. *Let $f(x,y)$ be a nonnegative n-order homogenous function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(x,x) : x \in \mathbb{R}_+\}$ which is three-time differentiable. Then $\mathcal{H}_{2f}(p)$ is log-convex (log-concave) in $p \in (0,1)$ if $J = (x-y)(xI)_x < (>) 0$, where $I = (\ln f)_{xy}$.*

For $p \neq q, p, q \in (0,1)$, define that

$$R_{2f}(p,q) := \left(\frac{\mathcal{H}_{2f}(p)}{\mathcal{H}_{2f}(q)} \right)^{\frac{1}{p-q}};$$

for $p \in (0,1), p \neq \frac{1}{2}$, define that

$$\begin{aligned} R_{2f}(p,p) &= \lim_{q \to p} R_{2f}(p,q) = \lim_{q \to p} \left(\frac{\mathcal{H}_{2f}(p)}{\mathcal{H}_{2f}(q)} \right)^{\frac{1}{p-q}} \\ &= \exp \lim_{q \to p} \frac{\ln \mathcal{H}_{2f}(p) - \ln \mathcal{H}_{2f}(q)}{p-q} = \exp \frac{d \ln \mathcal{H}_{2f}(p)}{dp} \\ &= \exp \frac{(T'(p) + T'(1-p))(2p-1) - 2\left[T(p) - T(1-p)\right]}{(2p-1)^2} \\ &= \exp \frac{(T'(p) + T'(1-p))(2p-1) - 2(2p-1)\ln \mathcal{H}_{2f}(p)}{(2p-1)^2} \\ &= \left(\frac{\exp T'(p) \exp T'(1-p)}{\mathcal{H}_{2f}^2(p)} \right)^{\frac{1}{2p-1}} \\ &= \left(\frac{G_{f,p} G_{f,1-p}}{\mathcal{H}_{2f}(p) \mathcal{H}_{2f}(1-p)} \right)^{\frac{1}{2p-1}}, \end{aligned}$$

where $G_{f,p}$ is defined by (1.3).

$$\begin{aligned} R_{2f}(\frac{1}{2}, \frac{1}{2}) &= \lim_{p \to \frac{1}{2}} R_{2f}(p, \frac{1}{2}) \\ &= \exp \left(\lim_{p \to \frac{1}{2}} \frac{d \ln \mathcal{H}_{2f}(p)}{dp} \right) \\ &= \exp \left(\lim_{p \to \frac{1}{2}} \frac{\left[T'(p) + T'(1-p)\right](2p-1) - 2\left[T(p) - T(1-p)\right]}{(2p-1)^2} \right) \\ &= \exp \lim_{p \to \frac{1}{2}} \frac{\left(T''(p) - T''(1-p)\right)(2p-1)}{4(2p-1)} \\ &= \exp(0) = 1 \end{aligned}$$

Thus from Theorem 2 and by properties of convex functions, we immediately get

Corollary 1. *That $R_{2f}(p,q)$ is strictly increasing (decreasing) in either p or q on $(0,1)$ if $J = (x-y)(xI)_x < (>) 0$, where*

$$R_{2f}(p,q) = \begin{cases} \left(\frac{\mathcal{H}_{2f}(p)}{\mathcal{H}_{2f}(q)} \right)^{\frac{1}{p-q}}, & p \neq q, p, q \in (0,1); \\ \left(\frac{G_{f,p} G_{f,1-p}}{\mathcal{H}_{2f}(p) \mathcal{H}_{2f}(1-p)} \right)^{\frac{1}{2p-1}}, & p = q \neq \frac{1}{2}, p, q \in (0,1); \\ 1, & p = q = \frac{1}{2}. \end{cases} \tag{2.8}$$

3. Proofs of the Main Results

To prove Theorem 1 and 2, we need certain lemmas of two-parameter homogeneous functions.

Lemma 1 ([22, Lemma 3, 4]). *Suppose that $f(x,y)$ is a positive n-order homogenous function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(x,x) : x \in \mathbb{R}_+\}$ which is three-time differentiable. Set $T(t) = \ln f(a^t, b^t)$, $a^t = x, b^t = y$ with $t \neq 0$, then*

$$T'(t) = \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} = \ln G_f^{\frac{1}{t}}(a^t,b^t), \tag{3.1}$$

where $G_f(x,y)$ is defined by (1.3);

$$T''(t) = -xyI\ln^2(b/a), \text{ where } I = (\ln f)_{xy}; \tag{3.2}$$

$$T'''(t) = -Ct^{-3}J, \text{ where } J = (x-y)(xI)_x, C = \frac{xy\ln^3(x/y)}{x-y} > 0. \tag{3.3}$$

Lemma 2 ([24, Property 4]). *Suppose that $f(x,y)$ is a positive symmetric n-order homogenous function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(x,x) : x \in \mathbb{R}_+\}$, then*

$$\mathcal{H}_f(t,-t) = G^n, \tag{3.4}$$

$$T(t) - T(-t) = 2nt\ln G, \tag{3.5}$$

$$T'(t) + T'(-t) = 2n\ln G, \tag{3.6}$$

$$T''(-t) = T''(t) \tag{3.7}$$

where $G = \sqrt{ab}$.

Remark 1. *If $f(1,1) := \lim_{x\to 1} f(x,1) > 0$, then define that $T'(0) := \lim_{t\to 0} T'(t) = n\ln G$. Thus (2.6) can be written as*

$$T'(t) + T'(-t) = 2T'(0). \tag{3.8}$$

Based on the above lemmas, we prove main results next.

A New Proof of Theroem 1. Without loss generality, we assume that $J = (x-y)(xI)_x < 0$. From (3.3) we see that $T''(t)$ is strictly increasing on $(0,\infty)$ and decreasing on $(-\infty, 0)$.

1) For $p \in (0,1)$. Since $\ln \mathcal{H}_{2f}(p) = \frac{T(p)-T(1-p)}{2p-1}$, a simply derivative calculation yields

$$\begin{aligned} \frac{d\ln \mathcal{H}_{2f}(p)}{dp} &= \frac{[T'(p)+T'(1-p)](2p-1) - 2[T(p)-T(1-p)]}{(2p-1)^2} \\ &= \frac{l(p)}{(2p-1)^2}, \end{aligned} \tag{3.9}$$

where

$$l(p) = (T'(p)+T'(1-p))(2p-1) - 2(T(p)-T(1-p)). \tag{3.10}$$

Note $l(\frac{1}{2}) = 0$ and

$$\begin{aligned} l'(p) &= (2p-1)(T''(p)-T''(1-p)) \\ &= (2p-1)^2 T'''(\xi), \end{aligned} \tag{3.11}$$

where $\xi = 1-p+\theta(2p-1), \theta \in (0,1)$. By Lemma 1 we see $T'''(\xi) > 0$ if $J = (x-y)(xI)_x < 0$ and $p \in (0,1)$, hence $l'(p) > 0$ on $(0,1)$. It follows that $l(p) < l(\frac{1}{2}) = 0$ for $p \in (0, \frac{1}{2})$ and $l(p) > l(\frac{1}{2}) = 0$ for $p \in (\frac{1}{2}, 1)$, and then

$$\frac{d \ln \mathcal{H}_{2f}(p)}{dp} = \frac{l(p)}{(2p-1)^2} \begin{cases} < 0, & p \in (0, \frac{1}{2}); \\ > 0, & p \in (\frac{1}{2}, 1), \end{cases}$$

which shows that $\mathcal{H}_{2f}(p)$ is strictly decreasing on $(0, \frac{1}{2})$ and increasing on $(\frac{1}{2}, 1)$.

2) For the symmetric function $f(x,y)$ and $p \in (-\infty,, \infty)$.

By part one of this Theorem, we understand that $l'(p) > 0$ for $p \in (0,1)$.

By (3.7) we have

$$\begin{aligned} l'(p) &= \left[T''(p) - T''(1-p)\right](2p-1) \\ &= \begin{cases} \left[T''(-p) - T''(1-p)\right](2p-1) > 0 & \text{for } p \in (-\infty, 0]; \\ \left[T''(p) - T''(p-1)\right](2p-1)) > 0 & \text{for } p \in [1, +\infty), \end{cases} \end{aligned}$$

which shows that $l'(p) > 0$ is always true for $p \in (-\infty,, \infty)$, i.e. that $l(p)$ is strictly increasing. It follows that $l(p) < l(\frac{1}{2}) = 0$ for $p \in (-\infty, \frac{1}{2})$ and $l(p) > l(\frac{1}{2}) = 0$ for $p \in (\frac{1}{2}, \infty)$, and then

$$\frac{d \ln \mathcal{H}_{2f}(p)}{dp} = \frac{l(p)}{(2p-1)^2} \begin{cases} < 0, & p \in (-\infty, \frac{1}{2}); \\ > 0, & p \in (\frac{1}{2}, \infty), \end{cases}$$

which completes the proof. ∎

Proof of Theorem 2. For $J = (x-y)(xI)_x < 0$. By (3.9)

$$\frac{d^2 \ln \mathcal{H}_{2f}(p)}{dp^2} = \frac{l'(p)(2p-1) - 4l(p)}{(2p-1)^3} = \frac{m(p)}{(2p-1)^3},$$

where

$$\begin{aligned} m(p) &= l'(p)(2p-1) - 4l(p) \\ &= \left(T''(p) - T''(1-p)\right)(2p-1)^2 - 4l(p). \end{aligned}$$

A derivative calculation leads to

$$m'(p) = \left(T'''(p) + T'''(1-p)\right)(2p-1)^2.$$

For $p \in (0,1)$, we have $T'''(p) > 0$ and $T'''(1-p) > 0$ if $J = (x-y)(xI)_x < 0$. This shows that $m'(p) > 0$ i.e. $m(p)$ is strictly increasing in $p \in (0,1)$. It follows that $m(p) < m(\frac{1}{2}) = 0$ for $p \in (0, \frac{1}{2})$ and $m(p) > m(\frac{1}{2}) = 0$ for $p \in (\frac{1}{2}, 1)$. Consequently, there always exists

$$\frac{d^2 \ln \mathcal{H}_{2f}(p)}{dp^2} = \frac{m(p)}{(2p-1)^3} > 0 \text{ for } p \in (0,1),$$

i.e. $\mathcal{H}_{2f}(p)$ is strictly log-convex in $p \in (0,1)$.

For $J = (x-y)(xI)_x > 0$ it can be proved in the same way.

The proof ends. ∎

4. Some applications

By Theorem 1 and 2, the monotonicity and log-convexity of $\mathcal{H}_{2f}(p)$ depend on the sign of $J = (x-y)(xI)_x$. In this section, by some straightforward computations, we present some conclusions about $\mathcal{H}_{2f}(p)$, where $f(x,y) = L(x,y), A(x,y), E(x,y)$ and $D(x,y)$.

(i) For $f(x,y) = L(x,y) = \dfrac{x-y}{\ln x - \ln y}$, where $x, y > 0$ with $x \neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2} \\ &= \frac{1}{xy(x-y)^2}\left((\sqrt{xy})^2 - L^2(x,y)\right) \\ J &= (x-y)(xI)_x = (x-y)\left(-\frac{x+y}{(x-y)^3} + \frac{2}{xy(\ln x - \ln y)^3}\right) \\ &= \frac{2}{xy(x-y)^2}\left(L^3(x,y) - \frac{x+y}{2}\left(\sqrt{xy}\right)^2\right). \end{aligned}$$

By the well-known inequalities $L(x,y) > \sqrt{xy}$ [17] and $L(x,y) > \left(\frac{x+y}{2}\right)^{\frac{1}{3}}\left(\sqrt{xy}\right)^{\frac{2}{3}}$ [10], we have $I < 0, J > 0$.

(ii) For $f(x,y) = A(x,y) = \dfrac{x+y}{2}$, where $x, y > 0$ with $x \neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = -\frac{1}{(x+y)^2} < 0, \\ J &= (x-y)(xI)_x = \frac{(x-y)^2}{(x+y)^3} > 0. \end{aligned}$$

(iii) For $f(x,y) = E(x,y) = e^{-1}\left(x^x/y^y\right)^{1/(x-y)}$, where $x, y > 0$ with $x \neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = \frac{1}{(x-y)^3}\left[2(x-y) - (x+y)(\ln x - \ln y)\right] \\ &= \frac{2(\ln x - \ln y)}{(x-y)^3}\left(L(x,y) - \frac{x+y}{2}\right) \\ J &= (x-y)(xI)_x = \frac{-3(x^2-y^2) + (x^2+4xy+y^2)(\ln x - \ln y)}{(x-y)^3} \\ &= -\frac{6(\ln x - \ln y)}{(x-y)^3}\left(\frac{x^2-y^2}{\ln x^2 - \ln y^2} - \frac{\frac{x^2+y^2}{2} + 2xy}{3}\right). \end{aligned}$$

By the well-known inequalities $L(x,y) < \frac{x+y}{2}$ [17] and $L(x,y) < \frac{\frac{x+y}{2} + 2\sqrt{xy}}{3}$ [7], we have $I < 0, J > 0$.

(iiii) For $f(x,y) = D(x,y) = |x-y|$, where $x, y > 0$ with $x \neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = \frac{1}{(x-y)^2} > 0, \\ J &= (x-y)(xI)_x = -\frac{x+y}{(x-y)^2} < 0. \end{aligned}$$

Notice that $L(x,y), A(x,y)$ and $E(x,y)$ are all symmetric with respect to x and y and $L(1,1) = A(1,1) = E(1,1) = 1$ but $D(1,1) = 0$, using Theorem 1 and 2, we easily obtain the following corollaries:

Corollary 2. *That $\mathcal{H}_{2L}(p;a,b), \mathcal{H}_{2A}(p;a,b)$ and $\mathcal{H}_{2E}(p;a,b)$ are strictly increasing in $p \in (-\infty, \frac{1}{2})$ and decreasing in $p \in (\frac{1}{2}, \infty)$ and log-concave in $p \in [0,1]$. That $R_{2L}(p), R_{2A}(p)$ and $R_{2E}(p)$ are strictly decreasing on $[0,1]$;*

Corollary 3. *That* $\mathcal{H}_{2D}(p;a,b)$ *is strictly decreasing in* $p \in (0, \frac{1}{2})$ *and increasing in* $p \in (\frac{1}{2}, 1)$ *and log-convex in* $(0,1)$. *That* $R_{2D}(p)$ *is strictly increasing in* $p \in (0,1)$.

Corollary 4. *For* $p_1, p_2 \in [\frac{1}{2}.1)$ *with* $p_1 < p_2$, *we have*

$$1 < \frac{\mathcal{H}_{2L}(p_1;a,b)}{\mathcal{H}_{2L}(p_2;a,b)} < e^{\frac{1}{L(p_2,1-p_2)}\,\frac{1}{L(p_1,1-p_1)}}. \tag{4.1}$$

Proof. For $p_1, p_2 \in [\frac{1}{2}.1)$ with $p_1 < p_2$, by Corollary 2 and 3, there are

$$\mathcal{H}_{2L}(p_1;a,b) > \mathcal{H}_{2L}(p_2;a,b), \tag{4.2}$$

$$\mathcal{H}_{2D}(p_1;a,b) < \mathcal{H}_{2D}(p_2;a,b). \tag{4.3}$$

Note

$$\mathcal{H}_{2D}(p;a,b) = \left(\frac{p}{1-p}\right)^{\frac{1}{2p-1}} \mathcal{H}_{2L}(p) = e^{\frac{1}{L(p,1-p)}}\mathcal{H}_{2L}(p), \tag{4.4}$$

then (4.3) can be changed as

$$e^{\frac{1}{L(p_1,1-p_1)}}\mathcal{H}_{2L}(p_1;a,b) < e^{\frac{1}{L(p_2,1-p_2)}}\mathcal{H}_{2L}(p_2;a,b). \tag{4.5}$$

From (4.5) and (4.2), we immediately get (4.1).

The proof ends. ∎

Corollary 5. *For* $p_1, p_2, q \in (0.1)$ *with* $p_1 < p_2$, *we have*

$$1 < \frac{R_{2L}(p_1,q)}{R_{2L}(p_2,q)} < e^{\frac{1}{p_2-q}\left(\frac{1}{L(p_2,1-p_2)} - \frac{1}{L(q,1-q)}\right) - \frac{1}{p_1-q}\left(\frac{1}{L(p_1,1-p_1)} - \frac{1}{L(q,1-q)}\right)}. \tag{4.6}$$

Proof. For $p, q \in (0.1)$, by (4.4) we have

$$R_{2D}(p,q) = \left(\frac{e^{\frac{1}{L(p,1-p)}}\mathcal{H}_{2L}(p)}{e^{\frac{1}{L(q,1-q)}}\mathcal{H}_{2L}(q)}\right)^{\frac{1}{p-q}} = e^{\frac{1}{p-q}\left(\frac{1}{L(p,1-p)} - \frac{1}{L(q,1-q)}\right)} R_{2L}(p,q). \tag{4.7}$$

If $p_1, p_2 \in (0.1)$ with $p_1 < p_2$, then by Corollary 2 and 3, there are

$$R_{2D}(p_1,q) < R_{2D}(p_2,q), \tag{4.8}$$

$$R_{2L}(p_1,q) > R_{2L}(p_2,q), \tag{4.9}$$

it follows from (4.7) and (4.8) that

$$e^{\frac{1}{p_1-q}\left(\frac{1}{L(p_1,1-p_1)} - \frac{1}{L(q,1-q)}\right)} R_{2L}(p_1,q) < e^{\frac{1}{p_2-q}\left(\frac{1}{L(p_2,1-p_2)} - \frac{1}{L(q,1-q)}\right)} R_{2L}(p_2,q). \tag{4.10}$$

Combining (4.9) and (4.10), we immediately get (4.6).

The proof ends. ∎

For $f(x,y) = L(x,y), A(x,y), E(x,y)$ and $D(x,y)$, according to the monotonicity of $\mathcal{H}_{2f}(p)$ we can get some interesting inequalities involving the logarithmic mean, exponential mean (identic mean), power mean and Heron mean etc. , for instance

[22, 24]:

$$\begin{aligned} G^{\frac{2}{3}}A^{\frac{1}{3}} &< \sqrt{GH} < G^{\frac{2}{5}} M_{\frac{1}{3}}^{\frac{1}{5}} M_{\frac{2}{3}}^{\frac{2}{5}} < L < M_{\frac{1}{5}}^{\frac{1}{3}} M_{\frac{2}{5}}^{\frac{2}{3}} \\ &< H_{\frac{1}{2}} < M_{\frac{1}{3}} < H_{\frac{2}{5}}^{2} M_{\frac{1}{5}}^{-1} < E_{\frac{1}{2}}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} G^{\frac{2}{3}} M_{2}^{\frac{2}{3}} A^{-\frac{1}{3}} &< G^{\frac{1}{2}} M_{\frac{3}{2}}^{\frac{3}{4}} M_{\frac{1}{2}}^{-\frac{1}{4}} < G^{\frac{2}{5}} M_{\frac{4}{3}}^{\frac{4}{5}} M_{\frac{1}{3}}^{-\frac{1}{5}} < A < M_{\frac{4}{5}}^{\frac{4}{3}} M_{\frac{1}{5}}^{-\frac{1}{3}} \\ &< M_{\frac{3}{4}}^{\frac{3}{2}} M_{\frac{1}{4}}^{-\frac{1}{2}} < M_{\frac{2}{3}}^{2} M_{\frac{1}{3}}^{-1} < M_{\frac{3}{5}}^{3} M_{\frac{2}{5}}^{-2} < Z_{\frac{1}{2}}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} G^{\frac{2}{3}} Z_{1}^{\frac{1}{3}} &< G^{\frac{1}{2}} E_{\frac{3}{2}}^{\frac{3}{4}} E_{\frac{1}{2}}^{-\frac{1}{4}} < G^{\frac{2}{5}} Z_{\frac{1}{3}}^{\frac{1}{5}} Z_{\frac{2}{3}}^{\frac{2}{5}} < E < Z_{\frac{1}{5}}^{\frac{1}{3}} Z_{\frac{2}{5}}^{\frac{2}{3}} \\ &< E_{\frac{3}{4}}^{\frac{3}{2}} E_{\frac{1}{4}}^{-\frac{1}{2}} < Z_{\frac{1}{3}} < E_{\frac{3}{5}}^{3} E_{\frac{2}{5}}^{-2} < Y_{\frac{1}{2}}, \end{aligned} \tag{4.13}$$

$$e^{2} E_{\frac{1}{2}} < \left(\tfrac{3}{2}\right)^{5} H_{\frac{2}{5}}^{2} M_{\frac{1}{5}}^{-1} < 2^{3} M_{\frac{1}{3}} < 3^{2} H_{\frac{1}{2}} < 4^{\frac{5}{3}} M_{\frac{1}{5}}^{\frac{1}{3}} M_{\frac{2}{5}}^{\frac{2}{3}}, \tag{4.14}$$

where

$$\begin{aligned} &A_p = A^{\frac{1}{p}}(a^p, b^p), A(a,b) = \tfrac{a+b}{2}; \\ &H_p = H^{\frac{1}{p}}(a^p, b^p), H(a,b) = \tfrac{a+\sqrt{ab}+b}{3}; \\ &E_p = E^{\frac{1}{p}}(a^p, b^p), E(a,b) = e^{-1}\left(b^b/a^a\right)^{1/(b-a)}; \\ &Z_p = Z^{\frac{1}{p}}(a^p, b^p), Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}; \\ &Y_p = Y^{\frac{1}{p}}(a^p, b^p), Y(a,b) = Ee^{1-G^2/L^2}. \end{aligned} \tag{4.15}$$

Using the log-convexity of $\mathcal{H}_{2f}(p)$ we can derive other new refined inequalities, where A_p,H_p,E_p, Z_p, Y_p are defined by (4.15).

Example 5. *For $f(x,y) = L(x,y)$, $R_{2L}(p,q)$ is strictly decreasing in either p or q on $[0,1]$.*

1) From $R_{2L}(\frac12,\frac23) > R_{2L}(\frac12,\frac34) > R_2(\frac12,1)$ i.e.

$$\left(\frac{A_{\frac13}}{E_{\frac12}}\right)^{6} > \left(\frac{H_{\frac12}}{E_{\frac12}}\right)^{4} > \left(\frac{L}{E_{\frac12}}\right)^{2}$$

it follows that

$$E_{\frac12} < A_{\frac13}^{3} H_{\frac12}^{-2}, \tag{4.16}$$

$$LE_{\frac12} < H_{\frac12}^{2}. \tag{4.17}$$

2) From $R_{2L}(\frac23,\frac12) > R_{2L}(\frac23,\frac23) > R_{2L}(\frac23,\frac34)$ i.e.

$$\left(\frac{A_{\frac13}}{E_{\frac12}}\right)^{6} > \left(\frac{E_{\frac13}E_{\frac23}}{A_{\frac13}^{2}}\right)^{3} > \left(\frac{H_{\frac12}}{A_{\frac13}}\right)^{12}$$

it follows that

$$H_{\frac12}^{2} A_{\frac13}^{-1} < \sqrt{E_{\frac13}E_{\frac23}} < A_{\frac13}^{2} E_{\frac12}^{-1}. \tag{4.18}$$

3) From $R_{2L}(\frac{3}{4},\frac{1}{2})>R_{2L}(\frac{3}{4},\frac{2}{3})>R_{2L}(\frac{3}{4},\frac{3}{4})>R_{2L}(\frac{3}{4},1)$ *i.e.*

$$\left(\frac{H_{\frac{1}{2}}}{E_{\frac{1}{2}}}\right)^4>\left(\frac{H_{\frac{1}{2}}}{A_{\frac{1}{3}}}\right)^{12}>\left(\frac{E_{\frac{1}{4}}E_{\frac{3}{4}}}{H_{\frac{1}{2}}^2}\right)^2>\left(\frac{L}{H_{\frac{1}{2}}}\right)^4$$

it follows that

$$L<\sqrt{E_{\frac{1}{4}}E_{\frac{3}{4}}}<H_{\frac{1}{2}}^4A_{\frac{1}{3}}^{-3}. \tag{4.19}$$

4) From $R_2(1,\frac{1}{2})>R_2(1,\frac{2}{3})>R_2(1,\frac{3}{4})>R_2(1,1)$ *i.e.*

$$\left(\frac{L}{E_{\frac{1}{2}}}\right)^2>\left(\frac{L}{A_{\frac{1}{3}}}\right)^3>\left(\frac{L}{H_{\frac{1}{2}}}\right)^4>\frac{EG}{L^2}$$

it follows that

$$LE_{\frac{1}{2}}^2 < A_{\frac{1}{3}}^3, \tag{4.20}$$

$$(\sqrt{EG})^{\frac{1}{3}}H_{\frac{1}{2}}^{\frac{2}{3}} < L<H_{\frac{1}{2}}^4A_{\frac{1}{3}}^{-3}. \tag{4.21}$$

Remark 2. *Since* $L<H_{\frac{1}{2}}<A_{\frac{1}{3}}<E_{\frac{1}{2}}$ *that second inequality of (4.21) is stronger than* $L<H_{\frac{1}{2}}<A_{\frac{1}{3}}$*; while first inequality gives a new lower bound of logarithmic mean* L*. Inequality (4.16) gives a new upper bound of exponential mean* E*.*

Note the corresponding relations of $f(x,y)=L(x,y)$ and $f(x,y)=A(x,y),E(x,y)$ (see Table 1), we can derive corresponding inequalities with (4.16)-(4.21).

f	$G_{f,p}$	$G_{f,0}$	$\mathcal{H}_{2f}(\frac{1}{2})$	$\mathcal{H}_{2f}(\frac{2}{3})$	$\mathcal{H}_{2f}(\frac{3}{4})$	$\mathcal{H}_{2f}(1)$
L	E_p	G	$E_{\frac{1}{2}}$	$A_{\frac{1}{3}}$	$H_{\frac{1}{2}}$	L
A	Z_p	G	$Z_{\frac{1}{2}}$	$A_{\frac{2}{3}}^2A_{\frac{1}{3}}^{-1}$	$A_{\frac{3}{4}}^{\frac{3}{2}}A_{\frac{1}{4}}^{-\frac{1}{2}}$	A
E	Y_p	G	$Y_{\frac{1}{2}}$	$Z_{\frac{1}{3}}$	$E_{\frac{3}{4}}^{\frac{3}{2}}E_{\frac{1}{4}}^{-\frac{1}{2}}$	E
D	$e^{\frac{1}{p}}E_p$	does not exist	$e^2E_{\frac{1}{2}}$	$8A_{\frac{1}{3}}$	$9H_{\frac{1}{2}}$	does not exist

Table 1. Comparison Table of L, A, E and D

Example 6. *For* $f(x,y)=A(x,y)$*,* $R_{2A}(p,q)$ *is strictly decreasing in either* p *or* q *on* $[0,1]$*, so we have*

$$Z_{\frac{1}{2}} < A_{\frac{2}{3}}^6A_{\frac{1}{3}}^{-3}A_{\frac{3}{4}}^{-3}A_{\frac{1}{4}}, \tag{4.22}$$

$$AZ_{\frac{1}{2}} < A_{\frac{3}{4}}^3A_{\frac{1}{4}}^{-1}. \tag{4.23}$$

$$A_{\frac{3}{4}}^3A_{\frac{1}{4}}^{-1}A_{\frac{2}{3}}^{-2}A_{\frac{1}{3}} < \sqrt{Z_{\frac{1}{3}}Z_{\frac{2}{3}}}<A_{\frac{2}{3}}^4A_{\frac{1}{3}}^{-2}Z_{\frac{1}{2}}^{-1} \tag{4.24}$$

$$A < \sqrt{Z_{\frac{1}{4}}Z_{\frac{3}{4}}}<A_{\frac{3}{4}}^6A_{\frac{1}{4}}^{-2}A_{\frac{2}{3}}^{-6}A_{\frac{1}{3}}^3 \tag{4.25}$$

$$AZ_{\frac{1}{2}}^2 < \left(A_{\frac{2}{3}}^2A_{\frac{1}{3}}^{-1}\right)^3, \tag{4.26}$$

$$(\sqrt{ZG})^{\frac{1}{3}}A_{\frac{3}{4}}A_{\frac{1}{4}}^{-\frac{1}{3}} < A<A_{\frac{3}{4}}^6A_{\frac{1}{4}}^{-2}A_{\frac{2}{3}}^{-6}A_{\frac{1}{3}}^3 \tag{4.27}$$

Example 7. *For $f(x,y)=E(x,y)$, note $E(a^2,b^2)/E(a,b)=Z(a,b)$* [20, Remark 4.1]*, $R_{2E}(p,q)$ is strictly decreasing in either p or q on $[0,1]$, so we have*

$$Y_{\frac{1}{2}} < Z_{\frac{1}{3}}^{3}E_{\frac{3}{4}}^{-3}E_{\frac{1}{4}}, \tag{4.28}$$

$$EY_{\frac{1}{2}} < E_{\frac{3}{4}}^{3}E_{\frac{1}{4}}^{-1}, \tag{4.29}$$

$$E_{\frac{3}{4}}^{3}E_{\frac{1}{4}}^{-1}Z_{\frac{1}{3}}^{-1} < \sqrt{Y_{\frac{1}{3}}Y_{\frac{2}{3}}} < Z_{\frac{1}{3}}^{2}Y_{\frac{1}{2}}^{-1}, \tag{4.30}$$

$$E < \sqrt{Y_{\frac{1}{4}}Y_{\frac{3}{4}}} < E_{\frac{3}{4}}^{6}E_{\frac{1}{4}}^{-2}Z_{\frac{1}{3}}^{-3}, \tag{4.31}$$

$$EY_{\frac{1}{2}}^{2} < Z_{\frac{1}{3}}^{3}, \tag{4.32}$$

$$(\sqrt{YG})^{\frac{1}{3}}E_{\frac{3}{4}}E_{\frac{1}{4}}^{-\frac{1}{3}} < E < E_{\frac{3}{4}}^{6}E_{\frac{1}{4}}^{-2}Z_{\frac{1}{3}}^{-3}. \tag{4.33}$$

Example 8. *For $f(x,y)=D(x,y)$, noticing $f(1,1)$ doesn't exist, $R_{2D}(p,q)$ is strictly increasing in either p or q on $(0,1)$, so we have the following inequalities corresponding with (4.16), (4.18) and (4.19)*

$$e^2E_{\frac{1}{2}} > 8^3A_{\frac{1}{3}}^{3}9^{-2}H_{\frac{1}{2}}^{-2} \tag{4.34}$$

$$9^2H_{\frac{1}{2}}^{2}8^{-1}A_{\frac{1}{3}}^{-1} > \sqrt{e^3E_{\frac{1}{3}}e^{\frac{3}{2}}E_{\frac{2}{3}}} > 8^2A_{\frac{1}{3}}^{2}e^{-2}E_{\frac{1}{2}}^{-1}. \tag{4.35}$$

$$\sqrt{e^4E_{\frac{1}{4}}e^{\frac{4}{3}}E_{\frac{3}{4}}} > 9^4H_{\frac{1}{2}}^{4}8^{-3}A_{\frac{1}{3}}^{-3}, \tag{4.36}$$

which can be rewritten as

$$E_{\frac{1}{2}} > (e^{-2}8^39^{-2})A_{\frac{1}{3}}^{3}H_{\frac{1}{2}}^{-2}, \tag{4.37}$$

$$(e^{-\frac{9}{4}}9^28^{-1})H_{\frac{1}{2}}^{2}A_{\frac{1}{3}}^{-1} > \sqrt{E_{\frac{1}{3}}E_{\frac{2}{3}}}, \tag{4.38}$$

$$\sqrt{E_{\frac{1}{4}}E_{\frac{3}{4}}} > (e^{-\frac{8}{3}}9^48^{-3})H_{\frac{1}{2}}^{4}A_{\frac{1}{3}}^{-3}. \tag{4.39}$$

Combining (4.16), (4.18) and (4.19) with (4.37), (4.38) and (4.39), we get

$$1 > E_{\frac{1}{2}}/(A_{\frac{1}{3}}^{3}H_{\frac{1}{2}}^{-2}) > e^{-2}8^39^{-2} \approx 0.855452655, \tag{4.40}$$

$$1 < \sqrt{E_{\frac{1}{3}}E_{\frac{2}{3}}}/(H_{\frac{1}{2}}^{2}A_{\frac{1}{3}}^{-1}) < e^{-\frac{9}{4}}9^28^{-1} \approx 1.067167149, \tag{4.41}$$

$$1 > \sqrt{E_{\frac{1}{4}}E_{\frac{3}{4}}}/.(H_{\frac{1}{2}}^{4}A_{\frac{1}{3}}^{-3}) > e^{-\frac{8}{3}}9^48^{-3} \approx 0.8903924291. \tag{4.42}$$

Remark 3. *The author gave a left estimation of exponential mean E by $A_{\frac{2}{3}}$, which is read as follows:*

$$1 > A_{\frac{2}{3}}/E > e/\sqrt{8} \approx 0.9610577571.$$

Replace $a^{\frac{1}{2}}, b^{\frac{1}{2}}$ with a, b in (4.40), then inequalities (4.40) can be rewritten as

$$1 < (A_{\frac{2}{3}}^{3}H^{-2})/E < \sqrt{e^28^{-3}9^2} \approx 1.081189977, \tag{4.43}$$

which shows that the relative error estimating exponential mean E by $A_{\frac{2}{3}}^{3}H^{-2}$ is approximate to 8%.

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