

**MONOTONICITY AND LOGARITHMIC CONVEXITY FOR A
CLASS OF ELEMENTARY FUNCTIONS INVOLVING THE
EXPONENTIAL FUNCTION**

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ABSTRACT. In this paper, the monotonicity and logarithmically convexity of the function $\frac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}$ are obtained, where $t \in \mathbb{R}$ and α and β are real numbers such that $\alpha \neq \beta$, $(\alpha, \beta) \neq (0, 1)$ and $(\alpha, \beta) \neq (1, 0)$.

1. INTRODUCTION

For real numbers α and β with $\alpha \neq \beta$, $(\alpha, \beta) \neq (0, 1)$ and $(\alpha, \beta) \neq (1, 0)$ and for $t \in \mathbb{R}$, let

$$q_{\alpha, \beta}(t) = \begin{cases} \frac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}, & t \neq 0, \\ \beta - \alpha, & t = 0. \end{cases} \quad (1)$$

In order to obtain the best bounds in Gautschi-Kershaw's inequalities, it was proved in [9] that the function $q_{\alpha, \beta}(t)$ is logarithmically convex in $(0, \infty)$ and logarithmically concave in $(-\infty, 0)$ if $\beta - \alpha > 1$ and is logarithmically concave in $(0, \infty)$ and logarithmically convex in $(-\infty, 0)$ if $0 < \beta - \alpha < 1$.

When ones study the logarithmically completely monotonic property of some functions involving Euler's gamma Γ function, the psi function ψ and the polygamma functions $\psi^{(i)}$ for $i \in \mathbb{N}$, the elementary function $q_{\alpha, \beta}(t)$ is encountered now and then. The so-called logarithmically completely monotonic function on an interval $I \subset \mathbb{R}$ is a positive function f which has derivatives of all orders on I and whose logarithm $\ln f$ satisfies $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$ for $k \in \mathbb{N}$ on I . The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. For more information on the class $\mathcal{L}[I]$, please refer to [1, 2, 3, 5, 6, 7, 8, 9] and the references therein.

The first aim of this paper is to research the monotonicity of the function $q_{\alpha, \beta}(t)$. The first main result of ours is the following Theorem 1 or Corollary 1.

Theorem 1. *The following conclusions present the monotonic properties of $q_{\alpha, \beta}(t)$.*

- (1) *The function $q_{\alpha, \beta}(t)$ is increasing in $(0, \infty)$ if either $1 \geq \alpha + \beta > 2\alpha + 1$ or $1 \leq \alpha + \beta < 2\alpha < \alpha + \beta + 1$ holds.*
- (2) *The function $q_{\alpha, \beta}(t)$ is decreasing in $(0, \infty)$ if either $1 \geq \alpha + \beta > 2\beta + 1$ or $1 \leq \alpha + \beta < 2\beta < \alpha + \beta + 1$ is valid.*

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- (3) The function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, 0)$ if either $2\alpha > \alpha + \beta + 1 \geq 2$ or $\alpha + \beta < 2\beta < \alpha + \beta + 1 \leq 2$ validates.
- (4) The function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, 0)$ if either $2\beta > \alpha + \beta + 1 \geq 2$ or $\alpha + \beta < 2\alpha < \alpha + \beta + 1 \leq 2$ sounds.
- (5) The function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, \infty)$ if and only if one of the following conditions holds:
- $\alpha = \beta + 1 > 1$,
 - $\alpha > \beta + 1 \geq 1$,
 - $\beta = \alpha + 1 < 1$,
 - $1 \geq \beta > \alpha + 1$,
 - $\alpha < \beta < \alpha + 1 \leq 1$,
 - $\beta + 1 \leq \alpha + \beta < 2\alpha < \alpha + \beta + 1$.
- (6) The function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, \infty)$ if and only if one of the following conditions holds:
- $\beta = \alpha + 1 > 1$,
 - $\beta > \alpha + 1 \geq 1$,
 - $\beta < \alpha < \beta + 1 \leq 1$,
 - $1 > \alpha = \beta + 1$,
 - $1 \geq \alpha > \beta + 1$,
 - $\alpha + 1 \leq \alpha + \beta < 2\beta < \alpha + \beta + 1$.

Remark 1. The (α, β) -domain where the function $q_{\alpha,\beta}(t)$ is monotonic in Theorem 1 can be described respectively by Figure 1 to Figure 6 below.

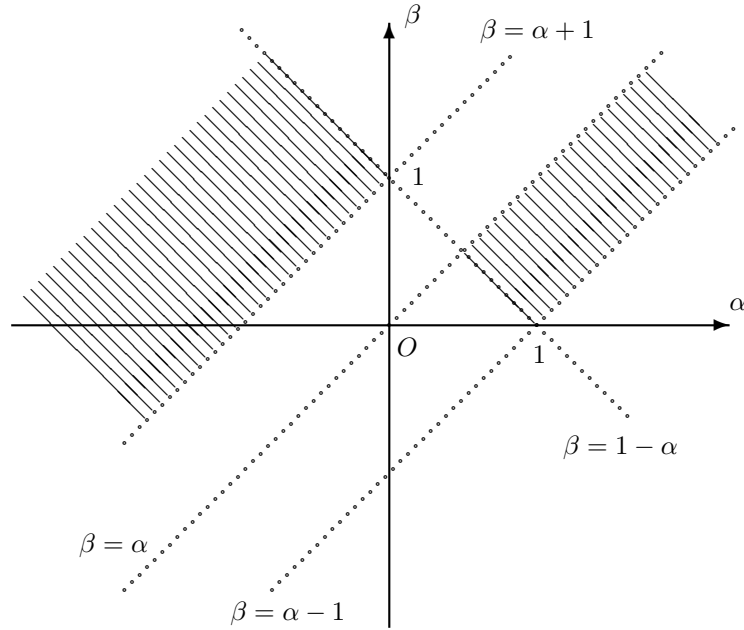


FIGURE 1. (α, β) -domain where the function $q_{\alpha,\beta}(t)$ is increasing in $(0, \infty)$ in Theorem 1

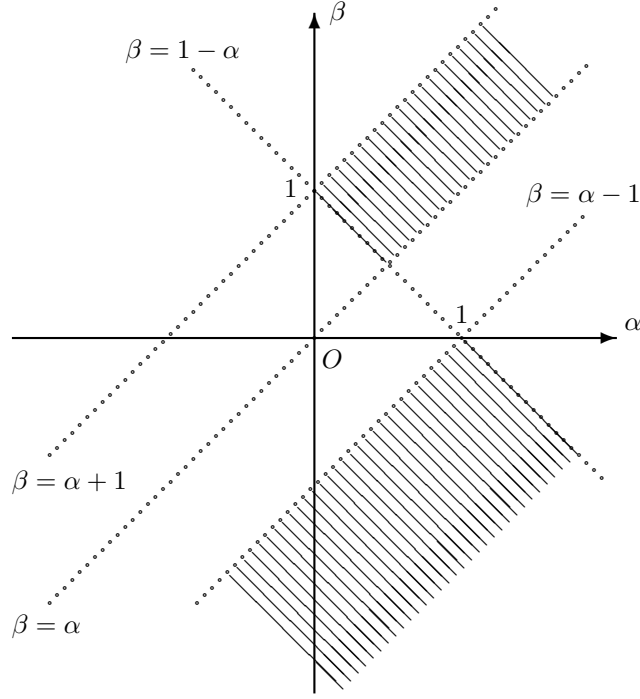


FIGURE 2. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is decreasing in $(0, \infty)$ in Theorem 1

Remark 2. Note that the (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is increasing (or decreasing) in $(0, \infty)$ (or in $(-\infty, 0)$) is an union where the function $q_{\alpha, \beta}(t)$ increases (or decreases) in either $(0, \infty)$ (or $(-\infty, 0)$) or $(-\infty, \infty)$. Therefore, Theorem 1 can be restated as the following Corollary 1.

Corollary 1. *The following conclusions describe the monotonic properties of $q_{\alpha, \beta}(t)$.*

- (1) *The function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, \infty)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) : \alpha > \beta \geq 0, \alpha \geq 1\} \cup \{(\alpha, \beta) : \alpha < \beta \leq 0\} \cup \{(\alpha, \beta) : \alpha \leq \beta - 1, 0 \leq \beta \leq 1\} \setminus \{(1, 0), (0, 1)\}$.*
- (2) *The function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, \infty)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) : \beta > \alpha \geq 0, \beta \geq 1\} \cup \{(\alpha, \beta) : \beta < \alpha \leq 0\} \cup \{(\alpha, \beta) : \beta \leq \alpha - 1, 0 \leq \alpha \leq 1\} \setminus \{(1, 0), (0, 1)\}$.*
- (3) *The function $q_{\alpha, \beta}(t)$ is increasing in $(0, \infty)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) : \alpha > \beta \geq \frac{1}{2}\} \cup \{(\alpha, \beta) : \alpha \geq 1 - \beta, 0 \leq \beta < \frac{1}{2}\} \cup \{(\alpha, \beta) : \alpha + 1 \leq \beta \leq 1 - \alpha, \alpha < 0\} \cup \{(\alpha, \beta) : \beta - 1 \leq \alpha < \beta \leq 0\} \setminus \{(1, 0)\}$.*
- (4) *The function $q_{\alpha, \beta}(t)$ is decreasing in $(0, \infty)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) : \beta \geq 1 - \alpha, \frac{1}{2} > \alpha \geq 0\} \cup \{(\alpha, \beta) : \beta > \alpha \geq \frac{1}{2}\} \cup \{(\alpha, \beta) : \beta < \alpha \leq 0\} \cup \{(\alpha, \beta) : \beta \leq \alpha - 1, 0 \leq \alpha \leq 1\} \cup \{(\alpha, \beta) : 1 \leq \alpha \leq 1 - \beta\} \setminus \{(1, 0), (0, 1)\}$.*
- (5) *The function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, 0)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) : 1 - \alpha \leq \beta < \alpha, \alpha \geq 1\} \cup \{(\alpha, \beta) : \alpha < \beta \leq 1, \alpha \leq 0\} \cup \{(\alpha, \beta) : \alpha < \beta \leq 1 - \alpha, 0 \leq \alpha < \frac{1}{2}\} \setminus \{(1, 0), (0, 1)\}$.*

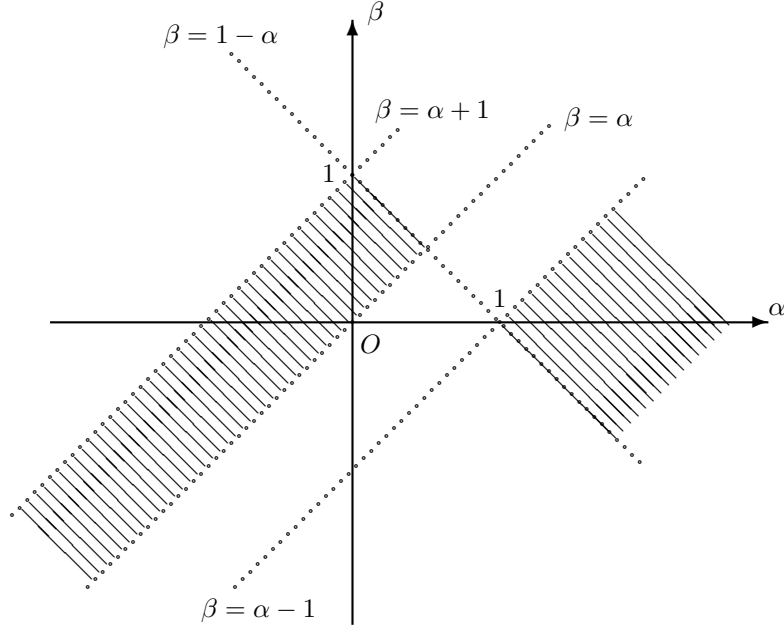


FIGURE 3. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, 0)$ in Theorem 1

- (6) The function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, 0)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) : 1 - \beta \leq \alpha < \beta, \beta \geq 1\} \cup \{(\alpha, \beta) : \beta < \alpha \leq \frac{1}{2}\} \cup \{(\alpha, \beta) : \beta \leq 1 - \alpha, \frac{1}{2} < \alpha \leq 1\} \setminus \{(1, 0), (0, 1)\}$.

Remark 3. The corresponding (α, β) -domains where the function $q_{\alpha, \beta}(t)$ is monotonic in Corollary 1 can be described respectively by Figure 5 to Figure 10 below.

The second aim of this paper is to reconsider the logarithmically convexity of the function $q_{\alpha, \beta}(t)$ by a very simpler approach than that in [9]. The second main result of ours is the following Theorem 2.

Theorem 2. The function $q_{\alpha, \beta}(t)$ in $(-\infty, \infty)$ is logarithmically convex if $\beta - \alpha > 1$ and logarithmically concave if $0 < \beta - \alpha < 1$.

Remark 4. Theorem 2 shows that the logarithmically convexity and logarithmically concavity in the interval $(-\infty, 0)$ of $q_{\alpha, \beta}(t)$ presented in [9] and mentioned at the beginning of this paper are wrong. However, this does not affect the correctness of the main results established in [9], since the wrong properties about $q_{\alpha, \beta}(t)$ in the interval $(-\infty, 0)$ are unuseful there luckily.

Remark 5. Recall that a r -times differentiable function $f(x) > 0$ is said to be r -log-convex (or r -log-concave) on an interval I with $r \geq 2$ if and only if $[\ln f(x)]^{(r)}$ exists and $[\ln f(x)]^{(r)} \geq 0$ (or $[\ln f(x)]^{(r)} \leq 0$) on I . In [4], the following conclusions are obtained: If $1 > \beta - \alpha > 0$, then $q_{\alpha, \beta}(t)$ is 3-log-convex in $(0, \infty)$ and 3-log-concave in $(-\infty, 0)$; if $\beta - \alpha > 1$, then $q_{\alpha, \beta}(t)$ is 3-log-concave in $(0, \infty)$ and 3-log-convex in $(-\infty, 0)$.

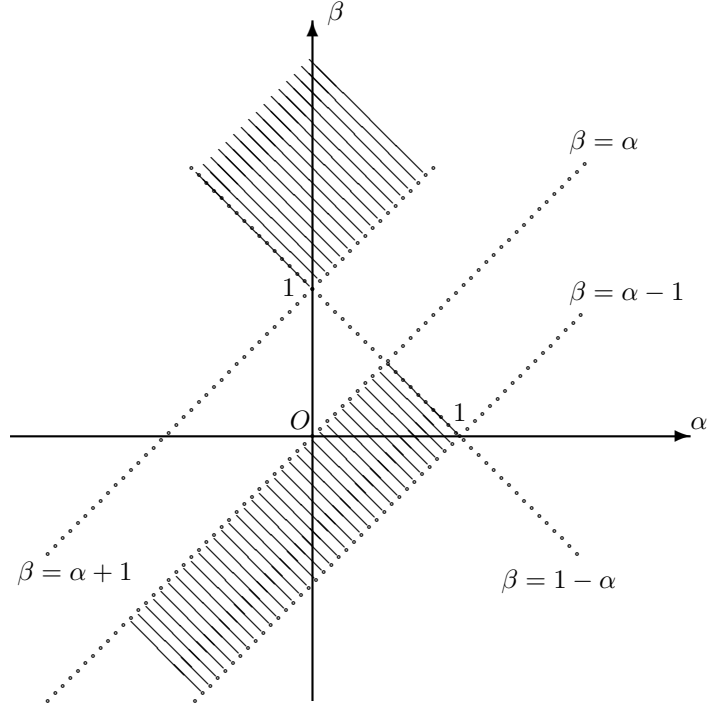


FIGURE 4. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, 0)$ in Theorem 1

2. PROOFS OF THEOREMS

Proof of Theorem 1. It is clear that the function $q_{\alpha, \beta}(t)$ can be rewritten as

$$q_{\alpha, \beta}(t) = \frac{\sinh \frac{(\beta - \alpha)t}{2}}{\sinh \frac{t}{2}} \exp \frac{(1 - \alpha - \beta)t}{2} \triangleq p_{\alpha, \beta} \left(\frac{t}{2} \right). \quad (2)$$

If $\alpha = \beta + 1$, then $q_{\alpha, \beta}(t) = -e^{-\beta t}$ is increasing for $\beta > 0$ and decreasing for $\beta < 0$ in $(-\infty, \infty)$. If $\alpha = \beta - 1$, then $q_{\alpha, \beta}(t) = e^{-\alpha t}$ is decreasing for $\alpha > 0$ and increasing for $\alpha < 0$ in $(-\infty, \infty)$.

For $|\alpha - \beta| \neq 1$, direct differentiation shows

$$p'_{\alpha, \beta}(t) = \frac{\sinh((\beta - \alpha)t)}{\sinh t} e^{(1 - \alpha - \beta)t} \varphi_{\alpha, \beta}(t),$$

where

$$\varphi_{\alpha, \beta}(t) = (\beta - \alpha) \coth((\beta - \alpha)t) - \coth t - \alpha - \beta + 1 \quad (3)$$

and

$$\begin{aligned} \varphi'_{\alpha, \beta}(t) &= \left(\frac{1}{\sinh t} \right)^2 - \left[\frac{\beta - \alpha}{\sinh((\beta - \alpha)t)} \right]^2 \\ &= \frac{1}{t^2} \left\{ \left(\frac{t}{\sinh t} \right)^2 - \left[\frac{(\beta - \alpha)t}{\sinh((\beta - \alpha)t)} \right]^2 \right\}. \end{aligned} \quad (4)$$

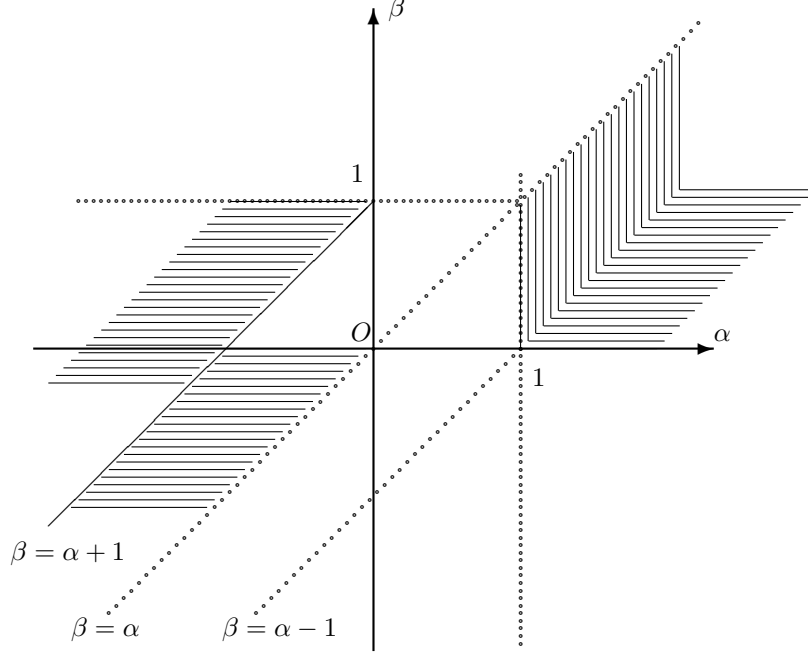


FIGURE 5. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, \infty)$ in Theorem 1 and Corollary 1

Since $\varphi'_{\alpha, \beta}(t) = \varphi'_{\alpha, \beta}(-t)$ and the function $\frac{t}{\sinh t} > 0$ is decreasing in $(0, \infty)$ and increasing in $(-\infty, 0)$, then $\varphi'_{\alpha, \beta}(t) \geq 0$ for $|\alpha - \beta| > 1$ and $\varphi'_{\alpha, \beta}(t) \leq 0$ for $0 < |\alpha - \beta| < 1$ in $(-\infty, \infty)$. This means that the function $\varphi_{\alpha, \beta}(t)$ is increasing for $|\alpha - \beta| > 1$ and decreasing for $0 < |\alpha - \beta| < 1$ in $(-\infty, \infty)$. It is not difficult to obtain $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) = 2 - \alpha - \beta - |\alpha - \beta|$, $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) = 1 - \alpha - \beta$ and $\lim_{t \rightarrow \infty} \varphi_{\alpha, \beta}(t) = |\alpha - \beta| - \alpha - \beta$.

1. If $\beta > \alpha + 1$, then $\beta - \alpha > 0$, $|\alpha - \beta| > 1$, $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) = 2(1 - \beta)$ and $\lim_{t \rightarrow \infty} \varphi_{\alpha, \beta}(t) = -2\alpha$. Further, if $\alpha \geq 0$, then $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(-\infty, \infty)$. Therefore, for $\beta > \alpha + 1 \geq 1$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\beta > \alpha + 1$ and $\beta \leq 1$, then $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) \geq 0$, $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(-\infty, \infty)$. Hence, for $1 \geq \beta > \alpha + 1$, the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, \infty)$.

If $\beta > \alpha + 1$ and $\alpha + \beta \leq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \geq 0$, $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(0, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(0, \infty)$. Consequently, for $2\alpha + 1 < \alpha + \beta \leq 1$, the function $q_{\alpha, \beta}(t)$ is increasing in $(0, \infty)$.

If $\beta > \alpha + 1$ and $\alpha + \beta \geq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \leq 0$, $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(-\infty, 0)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(-\infty, 0)$. Therefore, for $2\beta > \alpha + \beta + 1 \geq 2$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, 0)$.

2. If $\alpha < \beta < \alpha + 1$, then $\beta - \alpha > 0$ and $|\alpha - \beta| < 1$. Further, if $\alpha \leq 0$, then $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(-\infty, \infty)$. Accordingly, for $\alpha < \beta < \alpha + 1 \leq 1$, the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, \infty)$.

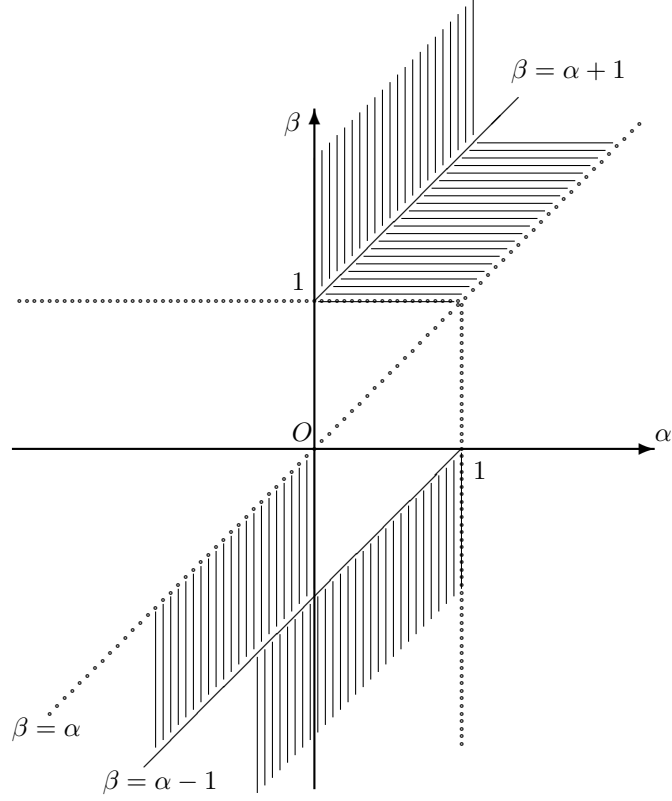


FIGURE 6. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, \infty)$ in Theorem 1 and Corollary 1

If $\alpha < \beta < \alpha + 1$ and $\beta \geq 1$, then $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) \leq 0$, $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(-\infty, \infty)$. Therefore, for $\alpha + 1 \leq \alpha + \beta < 2\beta < \alpha + \beta + 1$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\alpha < \beta < \alpha + 1$ and $\alpha + \beta \leq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \geq 0$, $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(-\infty, 0)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(-\infty, 0)$. As a result, for $\alpha + \beta < 2\beta < \alpha + \beta + 1 \leq 2$, the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, 0)$.

If $\alpha < \beta < \alpha + 1$ and $\alpha + \beta \geq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \leq 0$, $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(0, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(0, \infty)$. Consequently, for $1 \leq \alpha + \beta < 2\beta < \alpha + \beta + 1$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(0, \infty)$.

3. If $\alpha > \beta + 1$, then $\beta - \alpha < 0$, $|\alpha - \beta| > 1$, $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) = 2(1 - \alpha)$ and $\lim_{t \rightarrow \infty} \varphi_{\alpha, \beta}(t) = -2\beta$. Further, if $\beta \geq 0$, then $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(-\infty, \infty)$. Therefore, for $\alpha > \beta + 1 \geq 1$, the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, \infty)$.

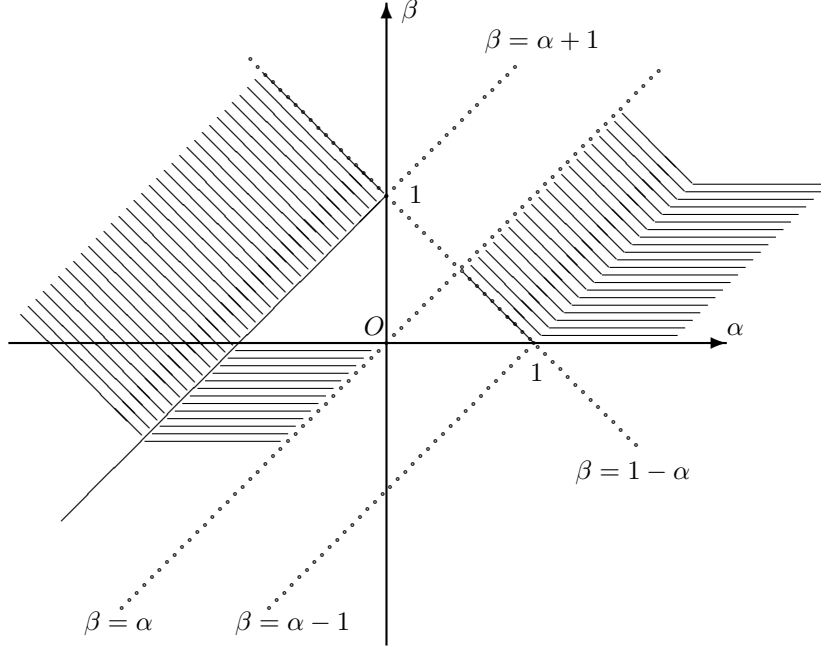


FIGURE 7. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is increasing in $(0, \infty)$ in Theorem 1

If $\alpha > \beta + 1$ and $\alpha \leq 1$, then $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) \geq 0$, $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(-\infty, \infty)$. Hence, for $1 \geq \alpha > \beta + 1$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\alpha > \beta + 1$ and $\alpha + \beta \leq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \geq 0$, $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(0, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(0, \infty)$. Accordingly, for $1 \geq \alpha + \beta > 2\beta + 1$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(0, \infty)$.

If $\alpha > \beta + 1$ and $\alpha + \beta \geq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \leq 0$, $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(-\infty, 0)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(-\infty, 0)$. Hence, for $2\alpha > \alpha + \beta + 1 \geq 2$, the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, 0)$.

4. If $\beta < \alpha < \beta + 1$, then $\beta - \alpha < 0$ and $|\alpha - \beta| < 1$. Further, if $\beta \leq 0$, then $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(-\infty, \infty)$. Therefore, for $\beta < \alpha < \beta + 1 \leq 1$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\beta < \alpha < \beta + 1$ and $\alpha \geq 1$, then $\lim_{t \rightarrow -\infty} \varphi_{\alpha, \beta}(t) \leq 0$, $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(-\infty, \infty)$. Accordingly, for $\beta + 1 \leq \alpha + \beta < 2\alpha < \alpha + \beta + 1$, the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, \infty)$.

If $\beta < \alpha < \beta + 1$ and $\alpha + \beta \leq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \geq 0$, $\varphi_{\alpha, \beta}(t) > 0$ and $p'_{\alpha, \beta}(t) < 0$ in $(-\infty, 0)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are decreasing in $(-\infty, 0)$. Consequently, for $\alpha + \beta < 2\alpha < \alpha + \beta + 1 \leq 2$, the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, 0)$.

If $\beta < \alpha < \beta + 1$ and $\alpha + \beta \geq 1$, then $\lim_{t \rightarrow 0} \varphi_{\alpha, \beta}(t) \leq 0$, $\varphi_{\alpha, \beta}(t) < 0$ and $p'_{\alpha, \beta}(t) > 0$ in $(0, \infty)$, and then $p_{\alpha, \beta}(t)$ and $q_{\alpha, \beta}(t)$ are increasing in $(0, \infty)$. As a

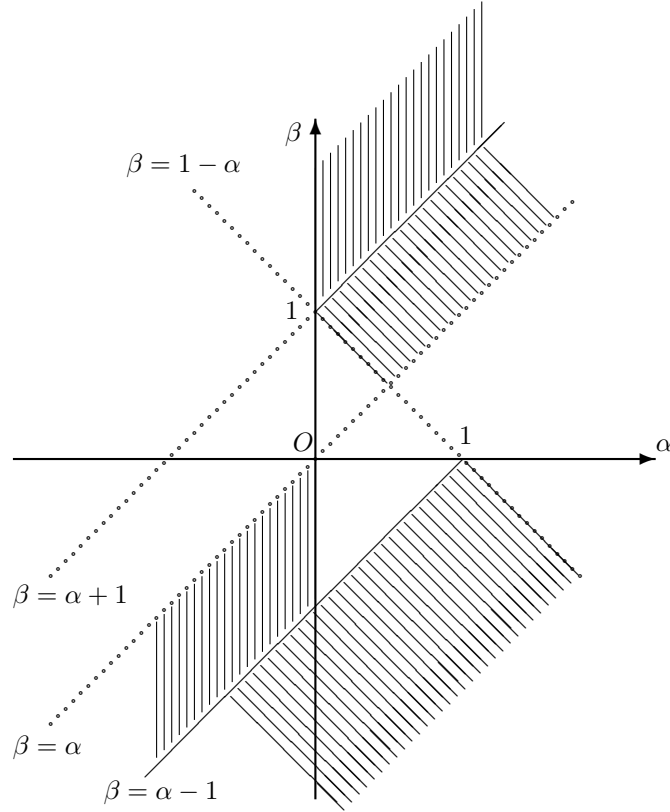


FIGURE 8. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is decreasing in $(0, \infty)$ in Theorem 1

result, for $1 \leq \alpha + \beta < 2\alpha < \alpha + \beta + 1$, the function $q_{\alpha, \beta}(t)$ is increasing in $(0, \infty)$. The proof of Theorem 1 is complete. \square

Proof of Theorem 2. For $\beta > \alpha$, the functions $q_{\alpha, \beta}(t)$ and $p_{\alpha, \beta}(t)$, related by (2), are positive. Taking logarithm of $p_{\alpha, \beta}(t)$ and differentiating yields

$$\begin{aligned} \ln p_{\alpha, \beta}(t) &= \ln \sinh((\beta - \alpha)t) - \ln \sinh t + (1 - \alpha - \beta)t, \\ [\ln p_{\alpha, \beta}(t)]' &= (\beta - \alpha) \coth((\beta - \alpha)t) - \coth t - \alpha - \beta + 1 = \varphi_{\alpha, \beta}(t), \end{aligned}$$

where $\varphi_{\alpha, \beta}(t)$ is defined by (3).

By the same argument as in the proof of Theorem 1 on page 5, it is easy to see that $\varphi'_{\alpha, \beta}(t) = [\ln p_{\alpha, \beta}(t)]'' \geq 0$ for $\beta - \alpha > 1$ and $\varphi'_{\alpha, \beta}(t) = [p_{\alpha, \beta}(t)]'' \leq 0$ for $0 < \beta - \alpha < 1$ in $(-\infty, \infty)$. This means that the function $p_{\alpha, \beta}(t) = q_{\alpha, \beta}(2t)$ is logarithmically convex for $\beta - \alpha > 1$ and logarithmically concave for $0 < \beta - \alpha < 1$ in the whole axis $(-\infty, \infty)$. The proof of Theorem 2 is complete. \square

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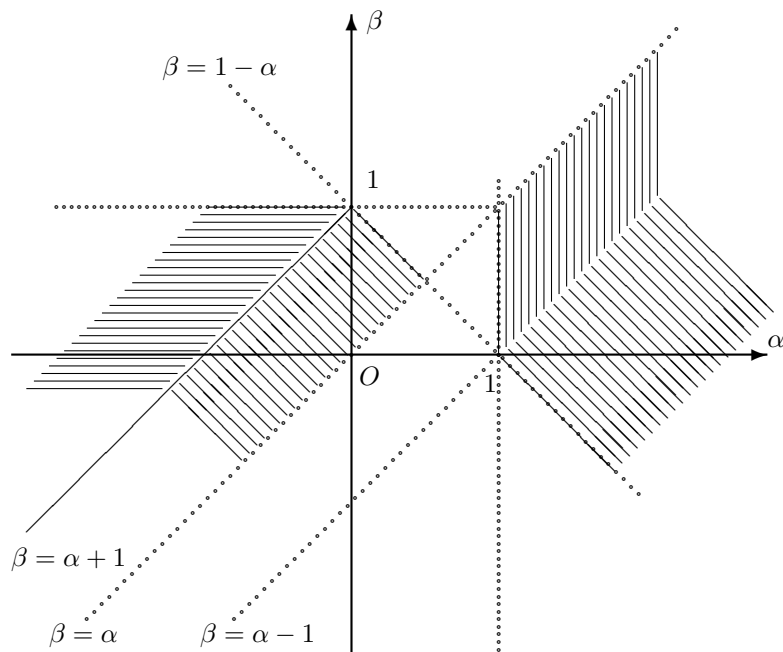


FIGURE 9. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is increasing in $(-\infty, 0)$ in Theorem 1

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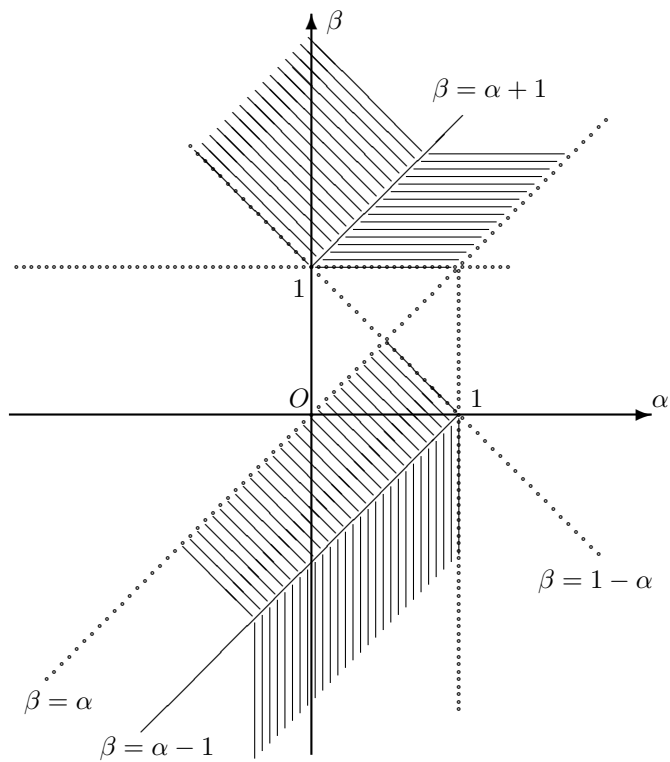


FIGURE 10. (α, β) -domain where the function $q_{\alpha, \beta}(t)$ is decreasing in $(-\infty, 0)$ in Theorem 1