APPROXIMATION OF THE SUM OF RECIPROCAL OF IMAGINARY PARTS OF ZETA ZEROS

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Abstract. In this paper, we approximate γ_n, where $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots$ are consecutive ordinates of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$; the Riemann zeta function. Then we obtain explicit bounds for the summation $\sum_{0<\gamma\leq T} \frac{1}{\gamma}$.

1. Introduction

The Riemann zeta-function is defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extended by analytic continuation to the complex plan with one singularity at $s = 1$; in fact a simple pole with residues 1. This was one of the results which B. Riemann obtained in his only paper on the theory of numbers [10], another one is functional equation which stated symmetrically as follows:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.1}$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ is a meromorphic function of the complex variable s, with simple poles at $s = 0, -1, -2, \cdots$ (see [8]). Riemann made a number of wonderful conjectures. For example, he guessed that the number $N(T)$ of zeros ρ of $\zeta(s)$ with $0 < \Im(\rho) \leq T$ and $0 \leq \Re(\rho) \leq 1$, satisfies the following relation:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

This conjecture of Riemann proved by H. von Mangoldt more than 30 years later [4, 7]. Some immediate corollaries of above approximate formula, which is known as Riemann-van Mangoldt formula, are

$$\mathcal{A}(T) = \sum_{0<\gamma\leq T} \frac{1}{\gamma} = O(\log^2 T),$$

and $\gamma_n \sim \frac{2\pi n}{\log n}$ when $n \to \infty$, where $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots$ are consecutive ordinates of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, which follow by partial summation from Riemann-van Mangoldt formula and using the obvious inequality $N(\gamma_n - 1) < n \leq N(\gamma_n + 1)$, respectively [7]. In this paper, we make some explicit approximation of γ_n and $\mathcal{A}(T)$.

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2. Approximation of γ_n

In 1941, Rosser [11] introduced the following approximation of $N(T)$:

$$|N(T) - F(T)| \leq R(T) \qquad (T \geq 2), \tag{2.1}$$

where

$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}, \tag{2.2}$$

and

$$R(T) = 0.137 \log T + 0.443 \log \log T + 1.588. \tag{2.3}$$

In this paper, using Rosser's result, we approximate γ_n, and then $\mathcal{A}(T)$, explicitly. Using (2.1) and $N(\gamma_n) = n$, we have:

$$(F - R)(\gamma_n) \leq n \leq (F + R)(\gamma_n).$$

Both of the functions $(F \pm R)(T)$ are increasing for $T \geq 14$, thus,

$$(F + R)^{-1}(n) \leq \gamma_n \leq (F - R)^{-1}(n),$$

holds for every $n \geq 1$. Unfortunately, finding an explicit formula for the inverses $(F \pm R)^{-1}(T)$ isn't possible and we must replace error term R by another one. Let

$$Y(T) = \frac{25}{147} T.$$

For every $T \geq 14$, we have $R(T) \leq Y(T)$, and the functions $(F \pm Y)(T)$ are increasing for $T \geq 18$. Since $\gamma_2 \simeq 21.02$, we obtain:

$$(F + Y)^{-1}(n) \leq \gamma_n \leq (F - Y)^{-1}(n) \qquad (n \geq 2).$$

Now, we are able to find inverses $(F \pm Y)^{-1}(T)$; considering Lambert W function $W(x)$, defined by $W(x)e^{W(x)} = x$ for $x \in [-e^{-1}, +\infty)$, for every $n \geq 2$ we yield that:

$$\frac{1}{4} \frac{(8n-7)\pi}{W\left(\frac{1}{8}(8n-7)e^{-1+\frac{50}{147}\pi}\right)} \leq \gamma_n \leq \frac{1}{4} \frac{(8n-7)\pi}{W\left(\frac{1}{8}(8n-7)e^{-1-\frac{50}{147}\pi}\right)}, \tag{2.4}$$

which holds also for $n = 1$. To make some explicit bounds, independent of Lambert W function, we use the following bounds

$$\log x - \log \log x < W(x) < \log x,$$

which the left hand side holds true for $x > 41.19$ and the right hand side holds true for $x > e$ [5]. Thus, we obtain:

$$\gamma_n < \frac{2\pi(n - \frac{7}{8})}{\log(n - \frac{7}{8}) - \log\left(\log(n - \frac{7}{8}) - (1 + \frac{50}{147}\pi)\right) - (1 + \frac{50}{147}\pi)}, \tag{2.5}$$

which holds for $(n - \frac{7}{8})e^{-(1+\frac{50}{147}\pi)} > 41.19$ or equivalently for $n > \frac{7}{8} + 41.19e^{1+\frac{50}{147}\pi} \approx 326.83$, and by computation for $13 \leq n \leq 326$, too. Also, we obtain:

$$\frac{2\pi(n - \frac{7}{8})}{\log(n - \frac{7}{8}) - (1 - \frac{50}{147}\pi)} < \gamma_n, \tag{2.6}$$

which holds for $(n - \frac{7}{8})e^{-(1-\frac{50}{147}\pi)} > e$ or equivalently for $n > \frac{7}{8} + e^{2-\frac{50}{147}\pi} \approx 3.41$, and by computation for $n = 1$ and $n = 3$, too.

3. Approximation of $\mathcal{A}(T)$

We note that:

$$\mathcal{A}(T) = \mathcal{G}(N) = \sum_{n=1}^{N} \frac{1}{\gamma_n},$$

in which

$$N = \max\{n : \ \gamma_n \leq T\} = N(T).$$

Now, we are ready to make explicit bounds for $\mathcal{A}(T)$.

3.1. **Upper Bound.**

Consider (2.6) and the following inequality[1]:

$$\mathcal{G}(N_0) < \frac{4}{\pi} \sum_{n=1}^{N_0} \frac{\log\left(\frac{1}{8}\,(8n-7)\, e^{-1+\frac{50}{147}\pi}\right)}{8n-7} \qquad (N_0 = 9996).$$

For every $N \geq N_0$, we have

$$\begin{aligned} \mathcal{G}(N) &< \frac{4}{\pi} \sum_{n=1}^{N} \frac{\log\left(\frac{1}{8}\,(8n-7)\, e^{-1+\frac{50}{147}\pi}\right)}{8n-7} \\ &= \frac{4}{\pi} \sum_{n=1}^{N} \frac{\log(8n-7)}{8n-7} + c_1 \Psi\left(N + \frac{1}{8}\right) - c_1 \Psi\left(\frac{1}{8}\right), \end{aligned}$$

where $c_1 = \frac{25}{147} - \frac{1+3\log 2}{2\pi} \approx -0.3200403161$, and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ with $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, is digamma function [8]. For every $x \geq 1$, it is known that

$$\log\left(x - \frac{1}{2}\right) < \Psi(x) \leq \log\left(x - 1 + e^{-c}\right), \tag{3.1}$$

where $c \simeq 0.5772156649$ is Euler constant [1]. In other hand, we have:

$$\sum_{n=1}^{N} \frac{\log(8n-7)}{8n-7} < \sum_{n=2}^{N} \frac{\log(8n)}{8(n-1)} = \frac{\log 8}{8} H(N-1) + \frac{1}{8} \sum_{n=2}^{N} \frac{\log n}{n-1},$$

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[1] We generate this numerical inequality, because the inequality (2.6) isn't true for $n = 2$. To compute the value of N_0, which is best possible value, we used numerical data concerning zeros of $\zeta(s)$, due to A. Odlyzko [9] and the following program in Maple software worksheet:

```
restart:
with(stats):
N:=9996:
x:=array(1..N):
fp:=fopen("zeros1.txt",READ):
g:=0:
for i from 1 by 1 to N do g:=g+1/describe[mean](fscanf(fp,"%f",x[i])) end do:
fclose(fp):
G(N)=g;
```

G.A. Pirayesh helped me to write above program, which I deem my duty to thank him for his kind helps.

where $H(N)=\sum_{n=1}^{N}\frac{1}{n}$ and for every $N\geq 1$, we have $H(N)\leq c+\log(N-1+e^{1-c})$ (see [1]). Also, we have:

$$\sum_{n=2}^{N}\frac{\log n}{n-1}<\int_{1}^{N}\frac{\log t}{t-1}dt=-\mathrm{dilog}(N),$$

where $\mathrm{dilog}(x)$ is Dilogarithm function, defined by $\mathrm{dilog}(x)=\int_1^x\frac{\log t}{1-t}dt$ for $x>0$ (see [16]). It is known that [6] for every $x>1$, the inequalities

$$\mathcal{D}(x,N)<\mathrm{dilog}(x)<\mathcal{D}(x,N)+\frac{1}{x^N}$$

holds true for all $N\in\mathbb{N}$, with

$$\mathcal{D}(x,N)=-\frac{1}{2}\log^2 x-\frac{\pi^2}{6}+\sum_{n=1}^{N}\frac{\frac{1}{n^2}+\frac{1}{n}\log x}{x^n}.$$

Therefore, we have

$$-\frac{1}{2}\log^2 x-\frac{\pi^2}{6}+\frac{1+\log x}{x}<\mathrm{dilog}(x), \tag{3.2}$$

and using this, we obtain

$$\mathcal{G}(N)<\frac{1}{4\pi}\log^2 N+\left(\frac{\log 8}{2\pi}+c_1\right)\log N+\left(\frac{c\log 8}{2\pi}+\frac{\pi}{12}-c_1\Psi\left(\frac{1}{8}\right)\right)+E_1(N),$$

where,

$$E_1(N)=\frac{\log 8}{2\pi}\log\left(1+\frac{e^{1-c}-2}{N}\right)+c_1\log\left(1-\frac{3}{8N}\right)-\frac{1+\log N}{2\pi N}<-\frac{\log N}{2\pi N}.$$

Thus

$$\mathcal{G}(N)<\frac{1}{4\pi}\log^2 N+c_2\log N+c_3-\frac{1}{2\pi}\frac{\log N}{N}, \tag{3.3}$$

for every $N\geq 9996$, with $c_2=\frac{\log 8}{2\pi}+c_1\approx 0.0109130841$ and $c_3=\frac{c\log 8}{2\pi}+\frac{\pi}{12}-c_1\Psi\left(\frac{1}{8}\right)\approx -2.231824968$. Also, it holds true for $4905\leq N\leq 9995$, by computation. Remembering $N=N(T)$, and using (2.1), we obtain the following explicit upper bound:

$$\begin{aligned}\mathcal{A}(T) &< \frac{1}{4\pi}\log^2\left(F(T)+R(T)\right)+c_2\log\left(F(T)+R(T)\right)\\ &+ c_3-\frac{1}{2\pi}\frac{\log\left(F(T)+R(T)\right)}{F(T)+R(T)} \qquad (N(T)\geq 4905).\end{aligned} \tag{3.4}$$

3.2. **Lower Bound.** Consider (2.5), which holds true for $n\geq 13$, and $\mathcal{G}(12)\approx 0.3731710458$. For every $N\geq 13$ we have

$$\begin{aligned}\mathcal{G}(N) &> \mathcal{G}(12)+\frac{4}{\pi}\sum_{n=13}^{N}\frac{\log(n-\frac{7}{8})-\log\left(\log(n-\frac{7}{8})-(1+\frac{50}{147}\pi)\right)-(1+\frac{50}{147}\pi)}{8n-7}\\ &= \frac{4}{\pi}\sum_{n=13}^{N}\left\{\frac{\log(8n-7)}{8n-7}-\frac{\log\left(147\log(n-\frac{7}{8})-147-50\pi\right)}{8n-7}\right\}\\ &+ c_4\Psi\left(N+\frac{1}{8}\right)+c_5,\end{aligned}$$

where

$$c_4 = \frac{4}{\pi}\left(-\frac{3}{8}\log 2 + \frac{1}{8}\log 147 - \frac{1}{8} - \frac{25}{588}\pi\right) \approx 0.1340756439,$$

and

$$c_5 = \mathcal{G}(12) - c_4\left(\frac{13236224754014816}{1220833367678925} + \Psi\left(\frac{1}{8}\right)\right) + \frac{6618112377007408}{1220833367678925\pi} \approx 1.769772.$$

Easily, we have:

$$\sum_{n=13}^{N} \frac{\log(8n-7)}{8n-7} = \frac{1}{8}\sum_{n=13}^{N} \frac{\log\left(n-\frac{7}{8}\right)}{n-\frac{7}{8}} + \frac{\log 8}{8}\sum_{n=13}^{N}\frac{1}{n-\frac{7}{8}},$$

and

$$\sum_{n=13}^{N} \frac{\log\left(n-\frac{7}{8}\right)}{n-\frac{7}{8}} > \int_{13-\frac{7}{8}}^{N+1-\frac{7}{8}} \frac{\log t}{t}dt = \frac{1}{2}\log^2\left(N+\frac{1}{8}\right) + c_6,$$

with

$$c_6 = -\frac{9}{2}\log^2 2 + 3\log 2\log 97 - \frac{1}{2}\log^2 97 \approx -3.113184782,$$

and

$$\sum_{n=13}^{N} \frac{1}{n-\frac{7}{8}} = \Psi\left(N+\frac{1}{8}\right) + c_7,$$

with

$$c_7 = -\left(\frac{13236224754014816}{1220833367678925} + \Psi\left(\frac{1}{8}\right)\right) \approx -2.453465877.$$

In other hand, we have:

$$\begin{aligned}
\sum_{n=13}^{N} \frac{\log\left(147\log(n-\frac{7}{8}) - 147 - 50\pi\right)}{8n-7} &< \sum_{n=13}^{N} \frac{\log\left(147\log(n-\frac{7}{8})\right)}{8n-7} \\
&= \frac{1}{8}\log 147\Psi\left(N+\frac{1}{8}\right) + c_8 \\
&+ \sum_{n=13}^{N} \frac{\log\left(\log(8n-7) - \log 8\right)}{8n-7},
\end{aligned}$$

where

$$c_8 = -\frac{1}{8}\log 147\left(\frac{13236224754014816}{1220833367678925} + \Psi\left(\frac{1}{8}\right)\right) \approx -1.530482008.$$

Also, we have:

$$\begin{aligned}
\sum_{n=13}^{N} \frac{\log\left(\log(8n-7) - \log 8\right)}{8n-7} &< \sum_{n=13}^{N} \frac{\log\log(8n-7)}{8n-7} < \int_{8(12)-7}^{8N-7} \frac{\log\log t}{t}dt \\
&= (\log\log(8N-7) - 1)\log(8N-7) + c_9,
\end{aligned}$$

where

$$c_9 = -\log\log 89\log 89 + \log 89 \approx -2.251270867.$$

Therefore, combining all of above inequalities and considering (3.2), for every $N \geq 13$, we obtain:

$$\begin{aligned} \mathcal{G}(N) &> \frac{1}{4\pi}\log^2\left(N+\frac{1}{8}\right) - \frac{4}{\pi}\left(\log\log(8N-7)-1\right)\log(8N-7) \\ &+ c_{10}\log\left(N-\frac{7}{8}+e^{-c}\right) + c_{11}, \end{aligned}$$

with

$$c_{10} = \frac{3\log 2 - \log 147}{2\pi} + c_4 \approx -0.3292229701,$$

and

$$c_{11} = \frac{c_6 + (3\log 2)c_7}{2\pi} - \frac{4(c_8+c_9)}{\pi} + c_5 \approx 5.277388010.$$

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