A CLASS OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS AND THE BEST BOUNDS IN THE FIRST KERSHAW'S DOUBLE INEQUALITY

FENG QI

Abstract. In the article, the logarithmically complete monotonicity of a class of functions involving the Euler's gamma function are proved, a class of the first Kershaw type double inequalities are established, and the first Kershaw's double inequality and Wendel's inequality are generalized, refined or extended. Moreover, an open problem is posed.

1. Introduction

It is well known that the classical Euler's gamma function Γ can be defined for $x>0$ as $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$. The digamma or psi function ψ is defined as the logarithmic derivative of Γ and $\psi^{(i)}$ for $i\in\mathbb{N}$ are called polygamma functions.

The ratio $\frac{\Gamma(s)}{\Gamma(r)}$ has been researched by many mathematicians in the past more than fifty years. In [32] J. Wendel gave for $0<b<1$ and $x>0$ the following double inequality

$$\left(\frac{x}{x+b}\right)^{1-b}\le\frac{\Gamma(x+b)}{x^b\Gamma(x)}\le 1. \tag{1}$$

W. Gautschi showed in [7] for $0<s<1$ and $n\in\mathbb{N}$ that

$$n^{1-s}<\frac{\Gamma(n+1)}{\Gamma(n+s)}<\exp\left[(1-s)\psi(n+1)\right]. \tag{2}$$

A strenghened upper bound was given by T. Erber in [6]:

$$\frac{\Gamma(n+1)}{\Gamma(n+s)}<\frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2},\quad 0<s<1,\quad n\in\mathbb{N}. \tag{3}$$

J. D. Kečkić and P. M. Vasić gave in [11] the inequalities below:

$$\frac{b^{b-1}}{a^{a-1}}\cdot e^{a-b}<\frac{\Gamma(b)}{\Gamma(a)}<\frac{b^{b-1/2}}{a^{a-1/2}}\cdot e^{a-b},\quad 0<a<b. \tag{4}$$

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The following closer bounds were proved for $0 < s < 1$ and $x \ge 1$ by D. Kershaw in [12]:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}, \tag{5}$$

$$\exp\left[(1-s)\psi\left(x+s^{1/2}\right)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{6}$$

Let s and t be nonnegative numbers, $\alpha = \min\{s,t\}$, and

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \ne t \\ e^{\psi(x+s)} - x, & s = t \end{cases} \tag{7}$$

in $x \in (-\alpha, \infty)$. In [4, 5, 13, 14, 20, 30], a monotonicity and convexity of $z_{s,t}(x)$ was obtained: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$. From this, the best bounds in the first Kershaw's double inequality (5) were deduced.

For a and b being two constants, as $x \to \infty$, the following asymptotic formula is given in [1, p. 257 and p. 259]:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right). \tag{8}$$

For recent development and more detailed information on this topic, please refer to, for example, [4, 5, 13, 14, 19, 20, 22, 24, 30] and the references therein.

Recall [2, 5, 15, 26] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \ge 0$, and that a function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I. For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I are denoted respectively by $\mathcal{C}[I]$ and $\mathcal{L}[I]$. In [2, 15, 25, 26, 28, 29], it has been proved that $\mathcal{L}[I] \subset \mathcal{C}[I]$. The well known Bernstein's Theorem [33, p. 161] states that $f \in \mathcal{C}[(0,\infty)]$ if and only if $f(x) = \int_0^\infty e^{-xs}\,\mathrm{d}\,\mu(s)$, where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. In [2, Theorem 1.1] and [8] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [10, Theorem 4.4]. For more information on the classes $\mathcal{C}[I]$ and $\mathcal{L}[I]$, please refer to [2, 15, 25, 26, 27, 28, 29] and the references therein.

For $x > 0$ and $a > 0$, let

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)} \qquad \text{and} \qquad f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)}, \tag{9}$$

where Γ is the classical Euler's gamma function. In [24], among other things, the logarithmically completely monotonic properties of the functions $h_a(x)$ and $f_a(x)$ are obtained:

(1) $\lim_{x\to0+} h_a(x) = \frac{\Gamma(a+1)}{a^a}$ and $\lim_{x\to\infty} h_a(x) = 1$ for any $a > 0$,
(2) $h_a(x) \in \mathcal{L}[(0,\infty)]$ if $0 < a < 1$,
(3) $[h_a(x)]^{-1} \in \mathcal{L}[(0,\infty)]$ if $a > 1$;

(4) $\lim_{x\to\infty} f_a(x) = 1$ for any $a \in (0,\infty)$,
(5) $f_a(x) \in \mathcal{L}[(0,\infty)]$ and $\lim_{x\to 0+} f_a(x) = \infty$ if $a > 1$,
(6) $[f_a(x)]^{-1} \in \mathcal{L}[(0,\infty)]$ and $\lim_{x\to 0+} f_a(x) = 0$ if $0 < a < 1$.

Observe that the functions $h_a(x)$ and $f_a(x)$ can be rewritten as

$$h_a(x) = (x+a)^{1-a}\frac{\Gamma(x+a)}{\Gamma(x+1)} \qquad \text{and} \qquad f_a(x) = x^{1-a}\frac{\Gamma(x+a)}{\Gamma(x+1)}. \tag{10}$$

In [3], the function $\frac{\Gamma(x+1)}{\Gamma(x+s)}\left(x+\frac{s}{2}\right)^{s-1}$ for $s \in (0,1)$ is proved to be completely monotonic in $(0,\infty)$.

These hint us to consider the logarithmically complete monotonicity of the function

$$H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{11}$$

for $x \in (-\rho,\infty)$, where a, b and c are real numbers and $\rho = \min\{a,b,c\}$.

The first main result of this paper is the following Theorem 1.

Theorem 1. *Let a, b and c be real numbers and $\rho = \min\{a,b,c\}$. Then*

(1) $H_{a,b,c}(x) \in \mathcal{L}[(-\rho,\infty)]$ *if* $(a,b,c) \in D_1(a,b,c)$, *where*

$$\begin{aligned} D_1(a,b,c) = &\left\{(a,b,c) : a+b \ge 1, c \le b < c + \frac{1}{2}\right\} \\ &\cup \left\{(a,b,c) : a > b \ge c + \frac{1}{2}\right\} \\ &\cup \{(a,b,c) : 2a+1 \le a+b \le 1, a < c\} \\ &\cup \{(a,b,c) : b-1 \le a < b \le c\} \\ &\setminus \{(a,b,c) : a = c+1, b = c\}. \end{aligned} \tag{12}$$

(2) $[H_{a,b,c}(x)]^{-1} \in \mathcal{L}[(-\rho,\infty)]$ *if* $(a,b,c) \in D_2(a,b,c)$, *where*

$$\begin{aligned} D_2(a,b,c) = &\left\{(a,b,c) : a+b \ge 1, c \le a < c + \frac{1}{2}\right\} \\ &\cup \left\{(a,b,c) : b > a \ge c + \frac{1}{2}\right\} \\ &\cup \{(a,b,c) : b < a \le c\} \\ &\cup \{(a,b,c) : b+1 \le a, c \le a \le c+1\} \\ &\cup \{(a,b,c) : b+c+1 \le a+b \le 1\} \\ &\setminus \{(a,b,c) : a = c+1, b = c\} \\ &\setminus \{(a,b,c) : b = c+1, a = c\}. \end{aligned} \tag{13}$$

As a direct consequence of the monotonicity of $H_{a,b,c}(x)$ and a generalization and a refinement of the first Kershaw's double inequality (5), the following Theorem 2, the second main result of this paper, is established.

Theorem 2. *Let a, b and c be real numbers, $\rho = \min\{a,b,c\}$ and δ be a constant greater than $-\rho$. If $(a,b,c) \in D_1(a,b,c)$, then inequality*

$$(x+c)^{a-b} < \frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{14}$$

holds in $x \in (-\rho, \infty)$ *and inequality*

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \le \frac{\Gamma(\delta+a)}{\Gamma(\delta+b)} \left(\frac{x+c}{\delta+c}\right)^{a-b} \tag{15}$$

sounds in $x \in [\delta, \infty)$. *If* $(a,b,c) \in D_2(a,b,c)$, *then inequalities* (14) *and* (15) *are reversed in* $(-\rho, \infty)$ *and* $[\delta, \infty)$ *respectively.*

Remark 1. Let us take $a = 1$ and $0 < b < 1$ in inequality (14). Then inequality

$$(x+c)^{1-b} < \frac{\Gamma(x+1)}{\Gamma(x+b)} \tag{16}$$

is valid in $(-\rho, \infty)$ for $(b,c) \in D_1(1,b,c) \cap \{(b,c) : 0 < b < 1\} = \{0 < b < 1, c \le b < 1\} \setminus \{(0,0)\}$. This implies that, in particular, inequality

$$(x+b)^{1-b} < \frac{\Gamma(x+1)}{\Gamma(x+b)} \tag{17}$$

is valid in $(-b, \infty)$ for $0 < b < 1$.

It is clear that inequality (17) not only refines the lower bound but also extends the range of the argument x of the left hand side in inequality (5).

Remark 2. Now let us take $a = 1$, $0 < b < 1$ and $\delta = 1$ in inequality (15). Then inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \le \frac{1}{\Gamma(1+b)} \left(\frac{x+c}{1+c}\right)^{1-b} \tag{18}$$

validates in $[1, \infty)$ for $(b,c) \in D_1(1,b,c) \cap \{(b,c) : 0 < b < 1\} \cap \{(b,c) : -\rho < 1\} = \{0 < b < 1, c \le b < 1\} \cap \{(b,c) : -\rho < 1\} \setminus \{(0,0)\} = \{(b,c) : 0 < b < 1, -1 < c \le b < 1\}$. In particular, for $0 < b < 1$, inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \le \frac{1}{\Gamma(1+b)} \left(\frac{x+b}{1+b}\right)^{1-b} \tag{19}$$

makes sense in $x \in [1, \infty)$.

Standard argument reveals that if

$$x \ge \frac{\bigl(1/2 - \sqrt{b+1/4}\,\bigr)(1+b)\sqrt[1-b]{\Gamma(1+b)} + 1}{(1+b)\sqrt[1-b]{\Gamma(1+b)} - 1} \triangleq \lambda(b) \tag{20}$$

then inequality (19) would be better than the right hand side of (5). It is easy to obtain that $\lim_{b\to 0+} \lambda(b) = \infty$ and

$$\lim_{b \to 1-} \lambda(b) = \frac{e + e^{\gamma} - \sqrt{5}\, e^{\gamma}}{2e^{\gamma} - e} = 0.6123686 \cdots < 1,$$

where $\gamma = 0.57721566\cdots$ is the Euler-Mascheroni's constant. This means that inequality (19) refines the right hand side of (5) if b is closer enough to 1 and that the upper bound in (19) is better than the one in (5) if x is larger enough.

Remark 3. The inequality (1) can be rewritten as

$$(x+b)^{1-b} \le \frac{\Gamma(x+1)}{\Gamma(x+b)} \le x^{1-b}. \tag{21}$$

It is easy to see that the range of the argument x in inequality (17) is larger than that in the left hand side of inequality (21).

Taking $a=1$, $0<b<1$ and $\delta=0$ in inequality (15) yields

$$\frac{\Gamma(x+1)}{\Gamma(x+b)}\le\frac{1}{\Gamma(b)}\left(\frac{x+c}{c}\right)^{1-b} \tag{22}$$

for $(b,c)\in D_1(1,b,c)\cap\{(b,c):0<b<1\}\cap\{(b,c):-\rho<0\}=\{0<b<1,c\le b<1\}\cap\{(b,c):-\rho<0\}\setminus\{(0,0)\}=\{(b,c):0<b<1,0<c\le b<1\}$. In particular, inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)}\le\frac{1}{\Gamma(b)}\left(\frac{x+b}{b}\right)^{1-b} \tag{23}$$

makes true in $[0,\infty)$ for $0<b<1$. When

$$x>\frac{1}{b\sqrt[1-b]{\Gamma(b)}-1}, \tag{24}$$

the upper bound in (23) is better than that in (21).

Remark 4. Since

$$[H_{a,b,c}(x)]^{1/(a-b)}=\frac{1}{x+c}\left[\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]^{1/(a-b)}=\frac{z_{b,a}(x)+x}{x+c} \tag{25}$$

and

$$z_{b,a}(x)=[H_{a,b,c}(x)]^{1/(a-b)}(x+c)-x, \tag{26}$$

the monotonicity and convexity of $z_{b,a}(x)$ and the logarithmically complete monotonicity of $H_{a,b,c}(x)$ are connected.

Remark 5. Equation (25) shows that $(1+b)\sqrt[1-b]{\Gamma(1+b)}$ in (20) and $b\sqrt[1-b]{\Gamma(b)}$ in (24) can be rewritten as $[H_{1,b,b}(1)]^{1/(b-1)}$ and $[H_{1,b,b}(0)]^{1/(b-1)}$ respectively. The graphs of these two functions, pictured by Mathematica 5.2, remind us that these two functions are increasing in $b\in(-1,\infty)$ and $b\in(0,\infty)$ respectively.

In [19], using some monotonicity results and inequalities of the generalized weighted mean values with two parameters in [9, 16, 17, 21, 31], it was verified, among other things, that the functions $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$ are increasing in both $r>0$ and $s>0$. In [30], it was showed that $\frac{1}{z_{s,t}(x)+1}\in\mathcal{C}[(-\alpha,\infty)]$.

Now it is natural to propose the following open problem: Let $\delta\ge0$, $\lambda\ge0$ and μ be real constants and $k\in\mathbb{N}$ such that $\mu>\lambda(2\delta)^{2k-1}$. For $x,y\in(-\delta,\infty)$, define

$$\Phi_{\delta,\lambda,\mu,k}(x,y)=\begin{cases}\dfrac{1}{\lambda(x+y)^{2k-1}+\mu}\left[\dfrac{\Gamma(\delta+x)}{\Gamma(\delta+y)}\right]^{1/(x-y)}, & x\ne y,\\ \dfrac{e^{\psi(\delta+y)}}{2\lambda y^{2k-1}+\mu}, & x=y.\end{cases} \tag{27}$$

What about the monotonicity, complete monotonicity, logarithmically complete monotonicity or Schur-convexity of the function $\Phi_{\delta,\lambda,\mu,k}(x,y)$?

2. Lemmas

In order to prove our main results, the following lemmas are necessary.

Lemma 1 ([1])**.** *For $x>0$ and $\omega>0$,*

$$\frac{1}{x^\omega}=\frac{1}{\Gamma(\omega)}\int_0^\infty t^{\omega-1}e^{-xt}\,\mathrm{d}\,t. \tag{28}$$

For $k\in\mathbb{N}$ and $x>0$,

$$\psi(x)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}\,t. \tag{29}$$

Lemma 2 ([18, 23])**.** *For real numbers α and β with $(\alpha,\beta)\notin\{(0,1),(1,0)\}$ and $\alpha\ne\beta$, let*

$$q_{\alpha,\beta}(t)=\begin{cases}\dfrac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t\ne0,\\ \beta-\alpha, & t=0.\end{cases} \tag{30}$$

(1) *The function $q_{\alpha,\beta}(t)$ is increasing in $(0,\infty)$ if and only if $(\alpha,\beta)\in D_1(\alpha,\beta)$, where*

$$\begin{aligned}D_1(\alpha,\beta)=&\left\{(\alpha,\beta):\alpha>\beta\ge\frac12\right\}\\ &\cup\left\{(\alpha,\beta):\alpha\ge1-\beta,0\le\beta<\frac12\right\}\\ &\cup\{(\alpha,\beta):\alpha+1\le\beta\le1-\alpha,\alpha<0\}\\ &\cup\{(\alpha,\beta):\beta-1\le\alpha<\beta\le0\}\\ &\setminus\{(1,0)\}.\end{aligned} \tag{31}$$

(2) *The function $q_{\alpha,\beta}(t)$ is decreasing in $(0,\infty)$ if and only if $(\alpha,\beta)\in D_2(\alpha,\beta)$, where*

$$\begin{aligned}D_2(\alpha,\beta)=&\left\{(\alpha,\beta):\beta\ge1-\alpha,\frac12>\alpha\ge0\right\}\\ &\cup\left\{(\alpha,\beta):\beta>\alpha\ge\frac12\right\}\\ &\cup\{(\alpha,\beta):\beta<\alpha\le0\}\\ &\cup\{(\alpha,\beta):\beta\le\alpha-1,0\le\alpha\le1\}\\ &\cup\{(\alpha,\beta):1\le\alpha\le1-\beta\}\\ &\setminus\{(1,0),(0,1)\}.\end{aligned} \tag{32}$$

Remark 6. The (α,β)-domains $D_1(\alpha,\beta)$ and $D_2(\alpha,\beta)$ defined in Lemma 2, where the function $q_{\alpha,\beta}(t)$ is increasingly or decreasingly monotonic in $(0,\infty)$, can be described respectively by Figure 1 and Figure 2 below.

Remark 7. In [18, 23, 30], the monotonicity, logarithmic convexity and 3-log-convexity of the function $q_{\alpha,\beta}(t)$ in either $(-\infty,0)$, $(0,\infty)$ or $(-\infty,\infty)$ have been investigated thoroughly.

3. Proofs of theorems

Proof of Theorem 1. By formulas (28) and (29), direct computation yields

$$\ln H_{a,b,c}(x)=(b-a)\ln(x+c)+\ln\Gamma(x+a)-\ln\Gamma(x+b),$$

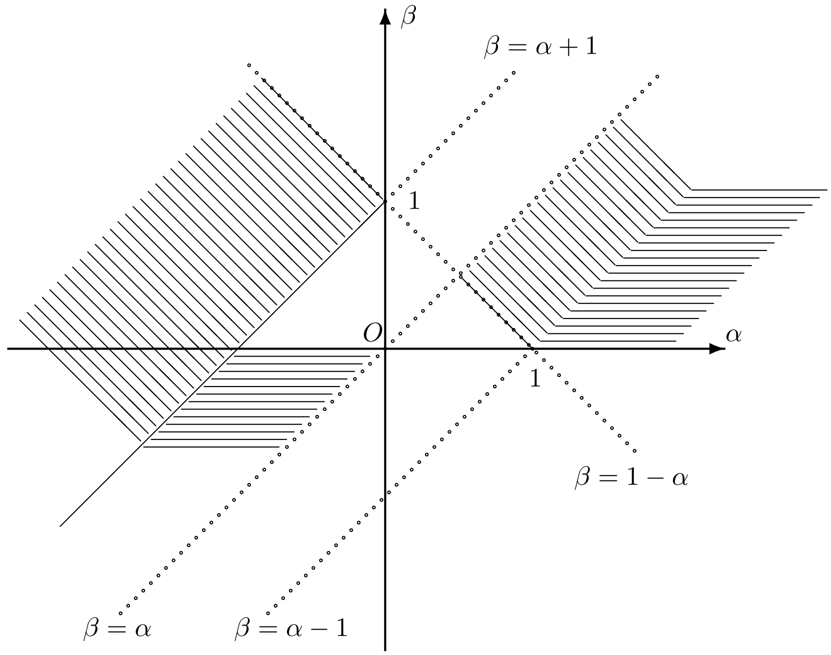

FIGURE 1. The (α,β)-domain $D_1(\alpha,\beta)$ in Lemma 2

$$
\begin{aligned}
[\ln H_{a,b,c}(x)]' &= \frac{b-a}{x+c} + \psi(x+a) - \psi(x+b) \\
&= \frac{b-a}{x+c} + \int_0^\infty \frac{e^{-bt}-e^{-at}}{1-e^{-t}} e^{-xt}\,\mathrm{d}t \\
&= -\int_0^\infty \left[\frac{e^{(c-a)t}-e^{(c-b)t}}{1-e^{-t}} + (a-b)\right] e^{-(x+c)t}\,\mathrm{d}t \\
&= -\int_0^\infty [q_{a-c,b-c}(t) + (a-b)] e^{-(x+c)t}\,\mathrm{d}t
\end{aligned}
$$

and, for $k \in \mathbb{N}$,

$$
(-1)^k [\ln H_{a,b,c}(x)]^{(k)} = \int_0^\infty [q_{a-c,b-c}(t) + (a-b)] t^{k-1} e^{-(x+c)t}\,\mathrm{d}t,
$$

where $q_{\alpha,\beta}(t)$ is defined by (30) in Lemma 2.

From $q_{\alpha,\beta}(0) = \beta - \alpha$ and $q_{a-c,b-c}(0) = b - a$, it is deduced that if $q_{a-c,b-c}(t)$ is increasing (or decreasing respectively) in $(0,\infty)$ then $q_{a-c,b-c}(t) + (a-b) \gtreqless 0$ in $t \in (0,\infty)$ and $(-1)^k [\ln H_{a,b,c}(x)]^{(k)} \gtreqless 0$ in $x \in (-\rho,\infty)$ for $k \in \mathbb{N}$. Combining this with Lemma 2 reveals that $H_{a,b,c}(x) \in \mathcal{L}[(-\rho,\infty)]$ if $(a-c, b-c) \in D_1(a-c, b-c)$ and $[H_{a,b,c}(x)]^{-1} \in \mathcal{L}[(-\rho,\infty)]$ if $(a-c, b-c) \in D_2(a-c, b-c)$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. By formula (8), it follows that

$$
H_{a,b,c}(x) = \left(1 + \frac{c}{x}\right)^{b-a} \left[x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)}\right]
$$

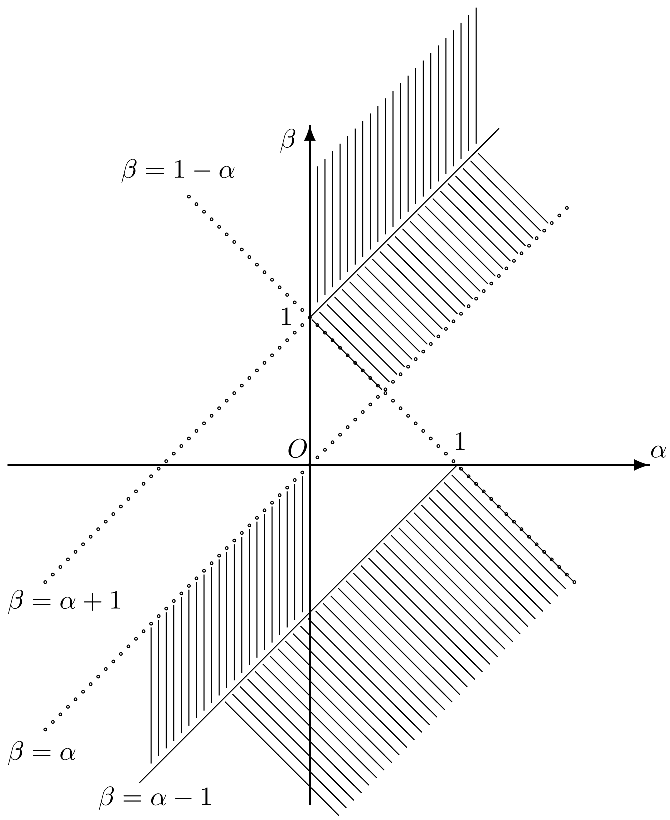

FIGURE 2. The (α, β)-domain $D_2(\alpha, \beta)$ in Lemma 2

$$= \left(1 + \frac{c}{x}\right)^{b-a}\left[1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right)\right]$$
$$\to 1$$

as $x \to \infty$ for all real numbers a, b and c.

If $(a, b, c) \in D_1(a, b, c)$, the function $H_{a,b,c}(x)$ is decreasing in $(-\rho, \infty)$ and $H_{a,b,c}(x) > \lim_{x\to\infty} = 1$ which can be rearranged as inequality (14). Further, if δ is a constant greater than $-\rho$, then

$$H_{a,b,c}(x) \le H_{a,b,c}(\delta) = (\delta + c)^{b-a}\frac{\Gamma(\delta + a)}{\Gamma(\delta + b)}$$

in $[\delta, \infty)$, which can be rewritten as (15) for $x \in [\delta, \infty)$.

If $(a, b, c) \in D_2(a, b, c)$ and δ is also a constant greater than $-\rho$, then the function $H_{a,b,c}(x)$ is increasing in $(-\rho, \infty)$, inequalities (14) and (15) are reversed respectively. The proof of Theorem 2 is complete. $\square$

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(F. Qi) College of Mathematics and Information Science, Henan Normal University, Xinxiang City, Henan Province, 453007, China; Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com, 316020821@qq.com`

URL: `http://rgmia.vu.edu.au/qi.html`