ON THE MONOTONICITY OF $\sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(-p,q)}$ AND ITS APPLICATIONS

ZHENHANG YANG

Abstract. Let $f(x,y)$ be a positive symmetric n-order homogenous function defined on $\mathbb{R}_+\times\mathbb{R}_+$ which is three-time differentiable. Then $\sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(-p,q)}$ is strictly increasing (decreasing) in p on $(0,\infty)$ if $J=(x-y)(xI)_x<(>)0$, where $I=(\ln f)_{xy}$. As applications, H. Alzer's inequalities are generalized and refined, some new inequalities for logarithmic mean, arithmetic mean and exponential mean are presented.

1. Introduction and Main Result

The generalized logarithmic mean between two positive numbers a and b is defined by

$$S_p(a,b)=\begin{cases}\left(\frac{b^p-a^p}{p(b-a)}\right)^{\frac{1}{p-1}}, & p\neq 0,1,b\neq a;\\ \frac{b-a}{\ln b-\ln a}, & p=0,b\neq a;\\ e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p=1,b\neq a;\\ a, & b=a.\end{cases}\tag{1.1}$$

Concerning $S_p(a,b)$, there are many results, in which A. Horst obtained the following inequalities [2]:

$$G(a,b)\leq\sqrt{S_p(a,b)S_{-p}(a,b)}\leq L(a,b),\tag{1.2}$$

where $G(a,b)=\sqrt{ab}, L(a,b)=(b-a)/\ln(b/a)(a\neq b), L(a,a)=a$.

Substituting $J_p(a,b)$ for $S_p(a,b)$, inequalities (1.2) still hold [3, 5], where

$$J_p(a,b)=\begin{cases}\frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & p\neq 0,-1,b\neq a;\\ \frac{a-b}{\ln a-\ln b}, & p=0,b\neq a;\\ \frac{ab(\ln a-\ln b)}{a-b}, & p=-1,b\neq a;\\ a & b=a.\end{cases}$$

On the other hand, the generalized logarithmic mean was extended into two-parameter homogeneous functions by the author [6]:

Definition 1. *Assume $f\colon\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+\cup\{0\}$ is an n-order homogeneous function of variables x and y which is continuous and exist first partial derivatives, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ with $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$.*

If $f(x,y)>0$ for $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ with $x\neq y$ and $f(x,x)=0$ for all $x\in\mathbb{R}_+$, then define that

$$\mathcal{H}_f(p,q;a,b)=\left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}\ (p\neq q,pq\neq 0),\tag{1.3}$$

$$\mathcal{H}_f(p,p;a,b)=\lim_{q\to p}\mathcal{H}_f(a,b;p,q)=G_{f,p}(p=q\neq 0),\tag{1.4}$$

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where

$$G_{f,p}=G_f^{\frac{1}{p}}(a^p,b^p),\quad G_f(x,y)=\exp\left(\frac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}\right), \tag{1.5}$$

$f_x(x,y)$ and $f_y(x,y)$ denote partial derivative with respect to first and second variable of $f(x,y)$ respectively.

If $f(x,y)>0$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ then define further

$$\mathcal{H}_f(p,0;a,b) = \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{\frac{1}{p}} (p\neq 0, q=0), \tag{1.6}$$

$$\mathcal{H}_f(0,q;a,b) = \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{\frac{1}{q}} (p=0, q\neq 0), \tag{1.7}$$

$$\mathcal{H}_f(0,0;a,b) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0)=a^{\frac{f_x(1,1)}{f(1,1)}}b^{\frac{f_y(1,1)}{f(1,1)}}(p=q=0). \tag{1.8}$$

Since $f(x,y)$ is a homogeneous function, $\mathcal{H}_f(p,q;a,b)$ is also one and called a homogeneous function with parameters p and q, and simply denote by $\mathcal{H}_f(p,q)$.

Put $q=p+1$ in the Definition 1, then $\mathcal{H}_f(p,p+1;a,b)$ is called a one-parameter homogeneous function [7]. Concerning $\mathcal{H}_f(p,p+1;a,b)$ the author obtained a generalization of (1.2), which is read as follows:

Theorem 1 ([7, Theorem 7])**.** *Let $f(x,y)$ be a positive symmetric n-order homogenous function defined on $\mathbb{R}_+\times\mathbb{R}$ which is three-time differentiable. Denote $\bar{\mathcal{H}}_{1f}(p)$ $=\mathcal{H}_{1f}(p)\mathcal{H}_{1f}(-p)$, then the function is strictly increasing (decreasing) in $p\in(0,+\infty)$ and strictly decreasing (increasing) in $p\in(-\infty,0)$ if*

$$J=(x-y)(xI)_x<(>)0,\ \textit{where}\ I=(\ln f)_{xy}. \tag{1.9}$$

By Theorem 1, we easily obtain the following

Corollary 1. *The conditions of $f(x,y)$ are the same as those of Theorem 1. Suppose that*

$$f(+0,1):=\lim_{x\to 0,x>0}f(x,1)=\gamma>0 \tag{1.10}$$

or

$$\gamma=0\ \textit{but exist}\ \delta>0\ \textit{such that}\ \lim_{x\to\infty}x^\delta f(e^{-x},1)=\lambda>0. \tag{1.11}$$

If 1.9 is true, then for $q>0$ the following inequalities hold:

$$G^n(a,b)<(>)\sqrt{\mathcal{H}_{1f}(p)\mathcal{H}_{1f}(-p)}<(>)\frac{f(a,b)}{f(1,1)} \tag{1.12}$$

In this paper, we will show a result that is similar to Theorem 1, which is also more generalized and refined than 1.2. It is stated as follows:

Theorem 2. *Let $f(x,y)$ be a positive symmetric n-order homogenous function defined on $\mathbb{R}_+\times\mathbb{R}_+$ which is three-time differentiable. Then $\bar{\mathcal{H}}_f(p)=\sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(-p,q)}$is strictly increasing (decreasing) in p on $(0,\infty)$ if 1.9 is true.*

Corollary 2. *$f(x,y)$ satisfies the conditions that are the same as those of Theorem 2, and (1.10) or (1.11). If $J=(x-y)(xI)_x<(>)0$, then for $q>0$ the following inequalities hold:*

$$G^n<(>)\sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(-p,q)}<(>)\mathcal{H}_f(0,q). \tag{1.13}$$

2. Proof of Main Results

For proving Theorem 2, we need certain lemmas of two-parameter homogeneous functions.

Lemma 1 ([8, Lemm 3, 4])**.** *Suppose that $f(x,y)$ is a positive n-order homogenous function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(x,x) : x \in \mathbb{R}_+\}$ which is three-time differentiable. Set $T(t) = \ln f(a^t, b^t)$, $a^t = x, b^t = y$ with $t \neq 0$, then*

$$T'(t) = \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} = \ln G_f^{\frac{1}{t}}(a^t,b^t), \tag{2.1}$$

where $G_f(x,y)$ is defined by (1.5);

$$T''(t) = -xyI\ln^2(b/a), \;\; where\ I = (\ln f)_{xy}; \tag{2.2}$$

$$T'''(t) = -Ct^{-3}J, \; where\ J = (x-y)(xI)_x, C = \frac{xy\ln^3(x/y)}{x-y} > 0. \tag{2.3}$$

Lemma 2 ([9, Property 4])**.** *Suppose that $f(x,y)$ is a positive symmetric of n-order homogenous function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \{(x,x) : x \in \mathbb{R}_+\}$, then*

$$\mathcal{H}_f(t,-t) = G^n, \tag{2.4}$$

$$T(t) - T(-t) = 2nt\ln G, \tag{2.5}$$

$$T'(t) + T'(-t) = 2n\ln G, \tag{2.6}$$

where $G = \sqrt{ab}$.

Remark 1. *If $f(1,1) := \lim_{x\to 1} f(x,1) > 0$, then define that $T'(0) := \lim_{t\to 0} T'(t) = n\ln G$. Thus (2.6) can be written as*

$$T'(t) + T'(-t) = 2T'(0). \tag{2.7}$$

Be equipped with Lemma 1 and 2, next let us prove main results.

Proof of Theorem 2. Since $f(x,y)$ is a symmetric n-order homogeneous function, we have

$$f(-p) = f(a^{-p}, b^{-p)} = (ab)^{-np} f(b^p, a^p) = (ab)^{-np} f(a^p, b^p),$$

hence

$$\begin{aligned}\mathcal{H}_f(-p,q) &= \left(\frac{f(a^{-p},b^{-p})}{f(a^q,b^q)}\right)^{\frac{1}{-p-q}} = \left((ab)^{-np}\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{-p-q}} \\ &= (ab)^{\frac{np}{p+q}}\left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}\,\frac{p-q}{-p-q}} = G^{\frac{2np}{p+q}}\mathcal{H}_f^{\frac{q-p}{p+q}}(p,q),\end{aligned}$$

and then

$$\bar{\mathcal{H}}_f(p) = \sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(-p,q)} = \mathcal{H}_f^{\frac{q}{p+q}}(p,q)(G^n)^{\frac{p}{p+q}}, \tag{2.8}$$

take logarithm to the above two sides,

$$\begin{aligned}\ln\bar{\mathcal{H}}_f(p) &= \frac{q}{p+q}\,\frac{T(p)-T(q)}{p-q} + \frac{p}{p+q}\ln G^n \\ &= \frac{q(T(p)-T(q)) + p(p-q)\ln G^n}{p^2-q^2}.\end{aligned} \tag{2.9}$$

A simple derivative calculation leads to

$$\begin{aligned}\frac{d\ln\bar{\mathcal{H}}_f(p)}{dp} &= \tfrac{1}{(p^2-q^2)^2}\left(qT'(p)+(2p-q)\ln G^n\right)(p^2-q^2)\\ &\quad -\tfrac{2p}{(p^2-q^2)^2}\left(q(T(p)-T(q))+p(p-q)\ln G^n\right)\\ &= \tfrac{p}{(p^2-q^2)^2}g(p),\end{aligned}$$

where

$$g(p)=q(p-\tfrac{q^2}{p})T'(p)-2q(T(p)-T(q))+(pq-2q^2+\tfrac{q^3}{p})\ln G^n.$$

It is easy to verify that $g(-q)=g(q)=0$ because

$$\begin{aligned}g(-q) &= q(-q-\tfrac{q^2}{-q})T'(-q)-2q(T(-q)-T(q))+(-qq-2q^2+\tfrac{q^3}{-q})\ln G^n\\ &= 2q(T(q)-T(-q))-4q^2\ln G^n\\ &= 4nq^2\ln G-4q^2\ln G^n=0 \text{ (by 2.5).}\end{aligned}$$

By a simple derivative operation, we get

$$\begin{aligned}g\prime(p) &= q(1+\tfrac{q^2}{p^2})T'(p)+q(p-\tfrac{q^2}{p})T^{''}(p)-2qT'(p)+(q-\tfrac{q^3}{p^2})\ln G^n\\ &= \frac{(p^2-q^2)q}{p}\left(T''(p)-\frac{1}{p}\left(T'(p)-\ln G^n\right)\right),\end{aligned}$$

note $\ln G^n=T'(0)$, then there must exist $\xi=\theta_1 p,\theta_1\in(0,1)$ and $\eta=\xi+\theta_2(p-\xi),\theta_2\in(\xi,p)$ by Mean value Theorem such that

$$\begin{aligned}g\prime(p) &= \frac{(p^2-q^2)q}{p}\left(T''(p)-\frac{1}{p}\left(T'(p)-T'(0)\right)\right)\\ &= \frac{(p^2-q^2)q}{p}\left(T''(p)-T''(\xi)\right)\\ &= \frac{(p^2-q^2)q}{p}(p-\xi)T^{'''}(\eta)\\ &= (1-\theta_1)(p^2-q^2)qT^{'''}(\eta).\end{aligned}$$

Since $\eta>0$ by (2.3) $T^{'''}(\eta)>0$ if $J<0$, then for $q>0$ there must be $g\prime(p)>0$ if $p>q$ and $g\prime(p)<0$ if $p<q$, it follows that $g(p)>g(q)=0$, and then we have $\frac{d\ln\bar{\mathcal{H}}_f(p)}{dp}>0$, i.e. $\bar{\mathcal{H}}_f(p)$ is strictly increasing; likewise for $q<0$ there must be $g(p)<g(-q)=0$ and then we have $\frac{d\ln\bar{\mathcal{H}}_f(p)}{dp}<0$, i.e. $\bar{\mathcal{H}}_f(p)$ is strictly decreasing.

In the same way, $\bar{\mathcal{H}}_f(p)$ is strictly decreasing (increasing) for $q>(<)0$ if $J>0$.

The proof is completed. ∎

Proof of Corollary 2. By Theorem 2, it is enough to calculate the limits of $\bar{\mathcal{H}}_f(p)$ at $p=0$ and $p=\infty$. In fact,

$$\lim_{p\to0}\mathcal{H}_f(p,q)=\lim_{p\to0}\mathcal{H}_f(-p,q)=\mathcal{H}_f(0,q),$$

hence

$$\lim_{p\to0}\bar{\mathcal{H}}_f(p)=\mathcal{H}_f(0,q).$$

On the other hand, assume $b>a$, then

$$\begin{aligned}\frac{T(p)-T(q)}{p-q} &= \frac{\ln f(a^p,b^p)-T(q)}{p-q}\\ &= \frac{np\ln b+\ln f((\frac{a}{b})^p,1)-T(q)}{p-q}.\end{aligned}$$

If $f(+0,1) := \lim_{x\to 0, x>0} f(x,1) = \gamma \neq 0$, then

$$\lim_{p\to\infty} \frac{T(p)-T(q)}{p-q} = n\ln b;$$

if $\gamma = 0$ but exist $\delta > 0$ such that $\lim_{x\to\infty} x^{\delta} f(e^{-x},1) = \lambda > 0$, then

$$\lim_{p\to\infty} \frac{T(p)-T(q)}{p-q} = \lim_{p\to\infty} \frac{np\ln b + \ln\big((p\ln(b/a))^{\delta} f(e^{-p\ln(b/a)},1)\big) - \ln(p\ln(b/a))^{\delta} - T(q)}{p-q} = n\ln b.$$

by (2.9)

$$\lim_{p\to\infty} \ln\bar{\mathcal{H}}_f(p) = \lim_{p\to\infty} \left(\frac{q}{p+q}\frac{T(p)-T(q)}{p-q} + \frac{p}{p+q}\ln G^n\right) = \ln G^n,$$

it follows that

$$\lim_{p\to\infty} \bar{\mathcal{H}}_f(p) = \exp(\lim_{p\to\infty} \ln\bar{\mathcal{H}}_f(p)) = \exp(\ln G^n) = G^n.$$

The proof ends. ∎

3. Some Applications

By Theorem 2, the monotonicity of $\bar{\mathcal{H}}_f(p) = \sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(-p,q)}$ depends on the sign of $J = (x-y)(xI)_x$. In this section, by some straightforward computations, we will present some conclusions about $\bar{\mathcal{H}}_f(p)$, where $f(x,y) = L(x,y), A(x,y), E(x,y)$.

(i) For $f(x,y) = L(x,y) = \dfrac{x-y}{\ln x - \ln y}$, where $x,y>0$ with $x\neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2} \\ &= \frac{1}{xy(x-y)^2}\left((\sqrt{xy})^2 - L^2(x,y)\right) \\ J &= (x-y)(xI)_x = (x-y)\left(-\frac{x+y}{(x-y)^3} + \frac{2}{xy(\ln x - \ln y)^3}\right) \\ &= \frac{2}{xy(x-y)^2}\left(L^3(x,y) - \frac{x+y}{2}(\sqrt{xy})^2\right). \end{aligned}$$

By the well-known inequalities $L(x,y) > \sqrt{xy}$ and $L(x,y) > \left(\frac{x+y}{2}\right)^{\frac{1}{3}}\left(\sqrt{xy}\right)^{\frac{2}{3}}$, we have $I<0, J>0$.

(ii) For $f(x,y) = A(x,y) = \dfrac{x+y}{2}$, where $x,y>0$ with $x\neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = -\frac{1}{(x+y)^2} < 0, \\ J &= (x-y)(xI)_x = \frac{(x-y)^2}{(x+y)^3} > 0. \end{aligned}$$

(iii) For $f(x,y) = E(x,y) = e^{-1}\left(x^x/y^y\right)^{1/(x-y)}$, where $x, y > 0$ with $x \neq y$, there are

$$\begin{aligned} I &= (\ln f)_{xy} = \frac{1}{(x-y)^3}\left[2(x-y) - (x+y)(\ln x - \ln y)\right] \\ &= \frac{2(\ln x - \ln y)}{(x-y)^3}\left(L(x,y) - \frac{x+y}{2}\right) \\ J &= (x-y)(xI)_x = \frac{-3(x^2-y^2) + (x^2+4xy+y^2)(\ln x - \ln y)}{(x-y)^3} \\ &= -\frac{6(\ln x - \ln y)}{(x-y)^3}\left(\frac{x^2-y^2}{\ln x^2 - \ln y^2} - \frac{\frac{x^2+y^2}{2} + 2xy}{3}\right). \end{aligned}$$

By the well-known inequalities $L(x,y) < \frac{x+y}{2}$ and $L(x,y) < \frac{\frac{x+y}{2}+2\sqrt{xy}}{3}$, we have $I < 0, J > 0$.

Notice that $L(x,y), A(x,y)$ and $E(x,y)$ are all symmetric with respect to x and y, and $L(1,1) = A(1,1) = E(1,1) = 1$, using Theorems 2 and (2.8), we immediately get the following corollaries:

Corollary 3. *For $q > 0$ that $\bar{\mathcal{H}}_f(p)$ is strictly decreasing in p on $(0,\infty)$ and the following inequalities hold:*

$$G < \sqrt{\mathcal{H}_L(p,q)\mathcal{H}_L(-p,q)} < L^{\frac{1}{q}}(a^q, b^q), \tag{3.1}$$

or

$$G < \mathcal{H}_L^{\frac{q}{p+q}}(p,q)G^{\frac{p}{p+q}} < L^{\frac{1}{q}}(a^q, b^q). \tag{3.2}$$

It is reversed if $q < 0$.

Remark 2. *Taking $q = 1$, we immediately get (1.2). It shows that Corollary 3 is a generalization and refinement of H. Alzer's inequalities.*

Taking $q = 1, p = 0, \frac{1}{2}, 1, 2$ in Corollary 3, we can get the following inequalities chain:

$$L > A_{\frac{1}{2}}^{\frac{2}{3}}G^{\frac{1}{3}} > \sqrt{EG} > A^{\frac{1}{3}}G^{\frac{2}{3}} > \cdots > G, \tag{3.3}$$

where $A_t = (\frac{a^t+b^t}{2})^{\frac{1}{t}}, E = e^{-1}(b^b/a^a)^{1/(b-a)} (b \neq a)$. Note $A_{\frac{1}{2}} = \frac{A+G}{2}$, then (3.3) can be rewritten as

$$L > (\tfrac{A+G}{2})^{\frac{2}{3}}G^{\frac{1}{3}} > \sqrt{EG} > A^{\frac{1}{3}}G^{\frac{2}{3}} > \cdots > G. \tag{3.4}$$

It follows from second and third inequality in (3.4) that

$$A^{\frac{2}{3}}G^{\frac{1}{3}} < E < (\tfrac{A+G}{2})^{\frac{4}{3}}G^{-\frac{1}{3}}. \tag{3.5}$$

Corollary 4. *For $q > 0$ that $\sqrt{\mathcal{H}_A(p,q)\mathcal{H}_A(-p,q)}$ is strictly decreasing in p on $(0,\infty)$ and the following inequalities hold:*

$$G < \sqrt{\mathcal{H}_A(p,q)\mathcal{H}_A(-p,q)} < A^{\frac{1}{q}}(a^q, b^q). \tag{3.6}$$

or

$$G < \mathcal{H}_A^{\frac{q}{p+q}}(p,q)G^{\frac{p}{p+q}} < A^{\frac{1}{q}}(a^q, b^q). \tag{3.7}$$

It is reversed if $q < 0$.

Taking $q = 1, p = 0, \frac{1}{2}, 1, 2$ in Corollary 4, we can get the following inequalities chain:

$$A > A^{\frac{4}{3}} A_{\frac{1}{2}}^{-\frac{2}{3}} G^{\frac{1}{3}} > \sqrt{ZG} > A_2^{\frac{2}{3}} A^{-\frac{1}{3}} G^{\frac{2}{3}} > \cdots > G, \tag{3.8}$$

where $Z = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$. It follows from second and third inequality in (3.8) that

$$A_2^{\frac{4}{3}} A^{-\frac{2}{3}} G^{\frac{1}{3}} < Z < A^{\frac{8}{3}} A_{\frac{1}{2}}^{-\frac{4}{3}} G^{-\frac{1}{3}}.$$

Corollary 5. *For $q > 0$ that $\sqrt{\mathcal{H}_E(p,q)\mathcal{H}_E(-p,q)}$ is strictly decreasing in p on $(0,\infty)$ and the following inequalities hold:*

$$G < \sqrt{\mathcal{H}_E(p,q)\mathcal{H}_E(-p,q)} < E^{\frac{1}{q}}(a^q, b^q), \tag{3.9}$$

or

$$G < \mathcal{H}_E^{\frac{q}{p+q}}(p,q) G^{\frac{p}{p+q}} < E^{\frac{1}{q}}(a^q, b^q). \tag{3.10}$$

where $E(a,b) = e^{-1}(b^b/a^a)^{1/(b-a)} (b \neq a)$. It is reversed if $q < 0$.

Taking $q = 1, p = 0, \frac{1}{2}, 1, 2$ in Corollary 5, notice

$$Z(a,b) = E(a^2, b^2)/E(a,b) \tag{3.11}$$

we can get the following inequalities chain:

$$E > Z_{\frac{1}{2}}^{\frac{2}{3}} G^{\frac{1}{3}} > \sqrt{YG} > Z^{\frac{1}{3}} G^{\frac{2}{3}} > \cdots > G, \tag{3.12}$$

where $Z_t = Z^{\frac{1}{t}}(a^t, b^t) = a^{\frac{a^t}{a^t+b^t}} b^{\frac{b^t}{a^t+b^t}}, Y = E e^{1-\frac{G^2}{L^2}}$.

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E-mail address: `yzhkm@163.comu`