A GEOMETRIC INEQUALITY FOR TWO INTERIOR POINTS OF THE TRIANGLE

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ABSTRACT. By using the basic inequality of the triangle and Stewart's theorem, a geometric inequality for two arbitrary interior points of a triangle is given, some known and new results are obtained. In the final, an open problem is posed.

1. INTRODUCTION AND RESULT

The following inequality (1.1) is well-known and it was obtained by M. S. Klamkin [1] in 1975:

Theorem 1.1. *Let $x, y, z \in R$. If P is an interior point of a triangle ABC, and $BC = a, CA = b, AB = c$, then*

$$(x+y+z)(xPA^2+yPB^2+zPC^2) \geqslant yza^2+zxb^2+xyc^2. \tag{1.1}$$

The equality in (1.1) holds if and only if the barycentric coordinates of point P is (x, y, z).

In this paper, we will obtain a new geometric inequality such is similar with Klamkin inequality above and involved two points inside the triangle. That is

Theorem 1.2. *If P, Q are two interior points of a triangle ABC, and (x, y, z) is the barycentric coordinates of point P (or Q) for triangle ABC, then*

$$(x+y+z)\left(xPA\cdot QA+yPB\cdot QB+zPC\cdot QC\right) \geqslant yza^2+zxb^2+xyc^2. \tag{1.2}$$

The equality in (1.2) holds if and only if two points P and Q are superposition.

2. THE PROOF OF THEOREM 1.2

Proof. Let $AQ\cap BC = D, BQ\cap CA = E, CQ\cap AB = F$, $AD = q_a, BE = q_b, CF = q_c, AQ = R_1, BQ = R_2, CQ = R_3$, and r_1, r_2, r_3 the distances from Q to the vertices of ΔABC. Then

$$ar_1 = x, br_2 = y, cr_3 = z, \tag{2.1}$$

$$q_a = \frac{x+y+z}{y+z}R_1, q_b = \frac{x+y+z}{z+x}R_2, q_c = \frac{x+y+z}{x+y}R_3, \tag{2.2}$$

where (x, y, z) is the barycentric coordinates of point Q for triangle ABC and r_1, r_2, r_3 are the distances from Q to BC, CA, AB.

From Stewart theorem, we have

$$AD^2 = q_a^2 = \frac{z}{y+z}b^2+\frac{y}{y+z}c^2-\frac{a^2yz}{(y+z)^2}, \tag{2.3}$$

and

$$PD^2 = \frac{z}{y+z}PC^2+\frac{y}{y+z}PB^2-\frac{a^2yz}{(y+z)^2}. \tag{2.4}$$

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For ΔAPD, using Cosine Law, we get

$$2q_a \cdot PA \geqslant 2q_a PA \cos\angle PAD = PA^2 + q_a^2 - PD^2. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5), we find

$$2x\,(y+z)\,q_a \cdot PA \geqslant x\,(y+z)\,PA^2 + zxb^2 + xyc^2 - zxPC^2 - xyPB^2, \tag{2.6}$$

with the equality holding if and only if the point P on the line AD.

Similarly, for $\Delta BPE, \Delta CPF$, we have

$$2y\,(z+x)\,q_b \cdot PB \geqslant y\,(z+x)\,PB^2 + xyc^2 + yza^2 - xyPA^2 - yzPC^2, \tag{2.7}$$

and

$$2z\,(x+y)\,q_c \cdot PC \geqslant z\,(x+y)\,PC^2 + yza^2 + zxb^2 - yzPB^2 - zxPA^2. \tag{2.8}$$

The equalities in (2.7) and (2.8) hold if and only if the point P on the lines BE and CF, respectively.

Summing (2.6),(2.7) and (2.8), we find

$$x\,(y+z)\,q_a \cdot PA + y\,(z+x)\,q_b \cdot PB + z\,(x+y)\,q_c \cdot PC \geqslant yza^2 + zxb^2 + xyc^2. \tag{2.9}$$

From (2.2), inequality (2.9) is equivalent to (1.2).

Utilizing the facts that

$$ar_1 + br_2 + cr_3 = 2\Delta,$$
$$ar_1 = x, br_2 = y, cr_3 = z, abc = 4R\Delta,$$

where Δ, R are the area and the circumradius of ΔABC, respectively. From (1.2), that is easily obtain

$$\frac{R_1 r_1}{bc} PA + \frac{R_2 r_2}{ca} PB + \frac{R_3 r_3}{ab} PC \geqslant 2R\left(\frac{r_2 r_3}{bc} + \frac{r_3 r_1}{ca} + \frac{r_1 r_2}{ab}\right). \tag{2.10}$$

Combining the condition of (2.6), (2.7) and (2.8) or inequality (1.1), we get

$$ar_1 R_1^2 + br_2 R_2^2 + cr_3 R_3^2 = 2R\,(ar_2 r_3 + br_3 r_1 + cr_1 r_2). \tag{2.11}$$

It is that equality in (1.2) holding if and only if two points P and Q are superposition. □

3. Some Applications

By the same way of Theorem 1.2, we can prove that

Corollary 3.1. *Let $P_1, P_2, \cdots, P_n, Q$ are $(n+1)$ interior points of ΔABC. Then*

$$\begin{aligned} &\frac{R_1 r_1}{bc}\,(PA_1 + PA_2 + \cdots + PA_n) + \frac{R_2 r_2}{ca}\,(PB_1 + PB_2 + \cdots + PB_n) + \\ &\frac{R_3 r_3}{ab}\,(PC_1 + PC_2 + \cdots + PC_n) \geqslant 2nR\left(\frac{r_2 r_3}{bc} + \frac{r_3 r_1}{ca} + \frac{r_1 r_2}{ab}\right). \end{aligned} \tag{3.1}$$

The equality in (3.1) holds if and only if $n+1$ points $P_1, P_2, \cdots, P_{n-1}$ and Q are superposition.

Combining Theorem 1.1 and Theorem 1.2, from Cauchy inequality, we can find that

Corollary 3.2. *If P, Q are two interior points of a triangle ABC, and (x, y, z) is the barycentric coordinates of point Q for triangle ABC, then*

$$\begin{aligned} &\left(xPA^2 + yPB^2 + zPC^2\right)\left(yza^2 + zxb^2 + xyc^2\right) \\ &\geqslant (x+y+z)\,(xPA \cdot QA + yPB \cdot QB + zPC \cdot QC)^2. \end{aligned} \tag{3.2}$$

The equality in (1.2) holds if and only if two points P and Q are superposition.

Using inequalities (1.2), (2.9), (2.10) and some known results, we can obtain some new inequalities. We prove only Corollary 3.3, and the proofs of Corollary 3.4-3.9 are left to the reader.

Corollary 3.3. *Let P is a point inside ΔABC. If m_a, m_b, m_c are the medians of ΔABC, then*

$$m_a PA + m_b PB + m_c PC \geqslant \frac{1}{2}\left(a^2 + b^2 + c^2\right). \tag{3.3}$$

The equality in (3.3) *holds if and only if p is the barycenter of ΔABC.*

Proof. Let Q and the barycenter G superpose. Then we have $R_1 = \frac{2}{3}m_a, R_2 = \frac{2}{3}m_b, R_3 = \frac{2}{3}m_c, r_1 = \frac{1}{3}h_a, r_2 = \frac{1}{3}h_b, r_3 = \frac{1}{3}h_c$ and $abc = 4R\Delta, 2Rh_a = bc, 2Rh_b = ca, 2Rh_c = ab$. From inequality (2.10), it follows (3.3). □

Corollary 3.4. *If $s = \frac{1}{2}(a + b + c)$, then*

$$PA \cos\frac{A}{2} + PB \cos\frac{B}{2} + PC \cos\frac{C}{2} \geqslant s, \tag{3.4}$$

with the equality holding if and only if P is the incenter of ΔABC.

Corollary 3.5. *If h_a, h_b, h_c are the altitudes of ΔABC, then*

$$\frac{m_a}{h_a}PA + \frac{m_b}{h_b}PB + \frac{m_c}{h_c}PC \geqslant 3R, \tag{3.5}$$

with the equality holding if and only if P is the center of gravity of ΔABC.

Corollary 3.6. *If P is an interior point of a triangle ABC, then*

$$cPA + aPB + bPC(or\ bPA + cPB + aPC) \geqslant \sqrt{b^2c^2 + c^2a^2 + a^2b^2}. \tag{3.6}$$

The equality in (3.6) *holds if and only if P is the positive (negative) Brocard point of ΔABC.*

Corollary 3.7. *If r_a, r_b, r_c are the radii of the escribed circle, r the inradius, and n_a, n_b, n_c Ceva lines through Nagel point of ΔABC, then*

$$\frac{n_a}{h_a r_a}PA + \frac{n_b}{h_b r_b}PB + \frac{n_c}{h_c r_c}PC \geqslant \frac{2R}{r} - 2, \tag{3.7}$$

with the equality holding if and only if P is Nagel point of ΔABC.

Corollary 3.8. *Let l_a, l_b, l_c be Ceva lines through Gergonne points of ΔABC. Then*

$$\frac{l_a}{h_a}PA + \frac{l_b}{h_b}PB + \frac{l_c}{h_c}PC \geqslant 2\left(R + r\right), \tag{3.8}$$

with the equality holding if and only if P is Gergonne of ΔABC.

Corollary 3.9. *If t_a, t_b, t_c are Ceva lines though the isotomic conjugate point of the incenter of ΔABC, then*

$$\left(b + c\right) t_a PA + \left(c + a\right) t_b PB + \left(a + b\right) t_c PC \geqslant a^3 + b^3 + c^3, \tag{3.9}$$

with the equality holding if and only if P is the isotomic conjugate point of the incenter of ΔABC.

4. An Open Problem

In the final, we pose an open problem.

Open Problem 4.1. *Let w_a, w_b, w_c be the bisectors of an angle and k_a, k_b, k_c the similar medians of ΔABC. For any interior P of ΔABC, then we have*

$$\frac{h_a}{w_a}PA + \frac{h_b}{w_b}PB + \frac{h_c}{w_c}PC \geqslant 6r, \tag{4.1}$$

and

$$\frac{h_a}{k_a}PA + \frac{h_b}{k_b}PB + \frac{h_c}{k_c}PC \geqslant 6r. \tag{4.2}$$

References

[1] M. S. Klamkin, *Geometric Inequalities via the Polar Moment of Inertia,* Math. Mag. **48**(1975), 44-46.

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