DIFFERENTIAL COMPLEXES WITH NONSMOOTHABLE COHOMOLOGY; OBSTRUCTIONS TO DE RHAM'S THEOREM

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Abstract. Given a locally integrable structure T' (e.g. a CR structure) on a smooth manifold $\mathcal{M}$, there is an associated differential complex of C^∞ differential (p, q)-forms and an associated differential complex of (p, q)-currents. The cohomology of the differential complex of (p, q)-currents is said to be smoothable if it is isomorphic to the cohomology of the corresponding complex of C^∞ differential (p, q)-forms. By the de Rham theorem, the cohomology of de Rham complex of currents on a real manifold and the cohomology of the $\overline{\partial}$-complex of (p, q)-currents on a complex manifold are both smoothable. The necessary conditions for smoothablitiy of diffential complexes associated with locally integrable structure are studied. Our results and methods for smoothablitiy are similar to ones for the vanishing of cohomology in [2], and [4].

Introduction

A theorem of de Rham states that for a smooth manifold $\mathcal{M}$, the cohomology $H^*(C^{(-\infty)}, \mathcal{M})$ of the complex of currents (i.e. forms with distributional coefficients) is isomorphic to the cohomology $H^*(C^\infty, \mathcal{M})$ of the complex of smooth differential forms. The smoothing of cohomology in this sense also holds for the Dolbeault $\overline{\partial}$-complexes on complex manifolds; $H^{p,*}(C^{(-\infty)}, \mathcal{M}) \simeq H^{p,*}(C^\infty, \mathcal{M})$, $p = 0, 1, 2, \ldots$(see e.g. [6]). The construction of $\overline{\partial}$-complexes on complex manifolds can be generalized to generate differential complexes of forms and currents of type (p, q) relative to any given involutive system of complex vector fields $L_1, L_2, \ldots, L_n$, $n \leq \dim \mathcal{M}$. The focus of this article is find necessary conditions for the smoothing of cohomology of such differential complexes. In fact, we consider obstructions to only the property that every cocycle is cohomologous to a smooth differential form.

Our results and methods are similar to the ones in [3] and [8]. The first step in the approach is to obtain the necessary *a priori estimate*. In our case, the estimates are in the form of $\left|\int f \wedge u\right| \leq \text{const}\,.\, \|f\| \left(\|u\|_{(-\mu)} + \|d'u\|\right)$, where f runs over all cocycle and u over forms of complementary degree, and $\|\cdot\|$ and $\|\cdot\|_{(-\mu)}$ are norms of C^k and Sobolev type, respectively. In the presence of a suitable topological obstruction of the underlying elliptic structure, one constructs a sequence of smooth cocycles $\{f_j\}$ and forms $\{u_j\}$ such that the limit of $\left|\int f_j \wedge u_j\right|$ as $j \to \infty$ is nonzero while the norms of f_j, $d'f_j$, u_j and $d'u_j$ approach zero, and thus violating the estimate.

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The paper is organized as follows. The background material in Section 1 is taken directly from [8]. In Section 3, we prove the *a priori* estimate mentioned above. The estimates of these type were first used by Hormander (see e.g. [7]) to study the solvability of differential and pseudodifferential operators. The version proved here is analogous to the one in [3]. The treatment of the top-degree forms in Section 5 is along the lines of Cordaro-Hounie[4]. In Section 4, we deal with the special case of locally integrable structures with nondegenerate Levi form. In Section 6, we follow the approach in [3], to treat the case of rigid structures of hypersurface type.

1. Preliminaries

Let M be a C^∞ manifold countable at infinity. For a real or complex vector bundle $\mathcal{E}$ over an open set U of M, we denote by $C^k(U,\mathcal{E})$, $0 \le k \le \infty$, the sheaf of C^k sections and by $C^{(-\infty)}(U,\mathcal{E})$, the sheaf of distributional sections over U, $C^k_c(U,\mathcal{E})$ and $C^{(-\infty)}_c(U,\mathcal{E})$ denote the corresponding subspaces of compactly supported sections. The notations $C^{(-\infty)}$, $C^{(-\infty)}_c$ are justified because these spaces are the duals of the topological vector spaces $C^\infty_c(U,\mathcal{E})$ and $C^\infty(U,\mathcal{E})$, respectively.

Let $\mathbb{C}T$M:= $\mathbb{C}\otimes T$M and $\mathbb{C}T^*$M:= $\mathbb{C}\otimes T^*$M denote the complexified tangent and cotangent bundles of $\mathcal{M}$.

Definition 1. *A subbundle T' of $\mathbb{C}T^*M$ is called a locally integrable structure if for every $x \in \mathcal{M}$ there exist C^∞ functions $Z_j : U \to \mathbb{C}$, $1 \le j \le m := \operatorname{rank} T'$, defined in a neighbourhood U of x such that $T'|_U$ is locally spanned by exact differentials $d\overline{Z}_1, d\overline{Z}_2..., d\overline{Z}_m$.*

Let $\mathcal{V} = (T')^\perp$ be the subbundle of the complexified tangent bundle $\mathbb{C}T$M orthogonal to T' under the natural duality between tangent and cotangent vectors. Then T' is locally integrable if and only if $\mathcal{V}$ is locally spanned by a system of complex vector fields $L_1, L_2, ..., L_m$ possessing the 'the first integrals' $Z_1, Z_2..., Z_m$ i.e.

$$L_i Z_j = 0, 1 \le j \le n, 1 \le i \le m, \text{ and } dZ_1 \wedge dZ_2 \wedge ... \wedge dZ_m \ne 0,$$

where $n := \operatorname{rank} \mathcal{V} = \dim \mathcal{M} - m$.

The above property implies that $\mathcal{V}$ is involutive i.e. closed under Frobenius bracket. Let $(x,\zeta) \in \mathbb{C}T^*$M The complex conjugate map

$$\mathbb{C}T^*\mathcal{M} \ni (x,\zeta) \longrightarrow (x,\overline{\zeta}) \in \mathbb{C}T^*\mathcal{M},$$

induces a bundle isomorphism of $T^{'}$ onto its conjugate $\overline{T}^{'}$. The set $T^0 = T^{'} \cap T^*$M is called the characteristic set of the locally integrable structure $T^{'}$. Observe $T^{'} \cap \overline{T}^{'} = T^0 \otimes \mathbb{C}$.

A locally integrable structure T' on a smooth manifold $\mathcal{M}$ is called (i) real if $\overline{T'} = T'$ or equivalently $\overline{\mathcal{V}} = \mathcal{V}$; (ii) complex if $\mathbb{C}T^*$M= $\overline{T'} \oplus T'$ or equivalently $\mathbb{C}T$M= $\overline{\mathcal{V}} \oplus \mathcal{V}$; (iii) elliptic if $T^0 = \{0\}$ or equivalently $\mathbb{C}T$M= $\overline{\mathcal{V}} + \mathcal{V}$ (iv) CR (Cauchy-Riemann) if $\mathbb{C}T^*$M= $\overline{T'} + T'$ or equivalently $\overline{\mathcal{V}} \cap \mathcal{V} = \{0\}$.

For a space $\mathcal{F}$ of functions or distributions and a vector bundle over $\mathcal{E}$ over M, we denote by $\mathcal{F}(U,\mathcal{E})$ the space of $\mathcal{F}$-sections of $\mathcal{E}$ over a subset U, and by $\mathcal{F}_c(U,\mathcal{E})$ the corresponding subspace of compactly supported sections. When $E = \Lambda^{p,q}$, the elements of $\mathcal{F}(U,\Lambda^{p,q})$ are called differential forms or currents of type (p,q).

For an integer $k \ge 0$, let $\Lambda^k(\mathbb{C}T^*\mathrm{M})$ denote the k-th exterior power of $\mathbb{C}T^*$M. For any pair of integers $p \ge 0,\ q \ge 0$ define $T^{'p,q}$ a vector subbundle of $\Lambda^{p+q}(\mathbb{C}T^*\mathrm{M})$

of all $(x,\zeta) \in$M$\times\Lambda^{p+q}(\mathbb{C}T^{*}M)$ such that ζ is the sum of exterior power $\xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_{p+q}$ in which at least p of the factors $\xi_j \in \mathbb{C}T^{*}$M belong to the fibre $T^{'}_x$. Let $d\psi_1, d\psi_2, \ldots, d\psi_m$ be the set of exact differentials that span $T^{'}$ over an open set U. If $\varsigma = \sum_j a_j(x)d\psi_j$ is any smooth section of $T^{'}$ over U then $d\varsigma = \sum_j da_j \wedge d\psi_j$ is clearly a section of $T^{'1,1}$. Similarly $dT^{\prime p,q} \subset T^{'p,q+1}$. Thus the exterior derivative induces a boundary operator denoted by d' on the quotient bundle $\Lambda^{p,q} := T^{\prime p,q}/T^{\prime p+1,q-1}$ (the bundle of (p,q) forms) i.e. $d'\Lambda^{p,q} \subset \Lambda^{p,q-1}$. The de Rham complex of $\mathcal{F}$ yields differential complexes for each $p = 0,1,2\ldots,m$,

$$(F^{p,*}(U))$$
$$\mathcal{F}(U,\Lambda^{p,0}) \xrightarrow{d'} \cdots \xrightarrow{d'} \mathcal{F}(U,\Lambda^{p,q}) \xrightarrow{d'} \mathcal{F}(U,\Lambda^{p,q+1}) \xrightarrow{d'} \cdots \xrightarrow{d'} \mathcal{F}(U,\Lambda^{p,n}).$$

Since $\operatorname{supp} d^{'p,q}u \subset \operatorname{supp} u$, the corresponding subcomplex $\mathcal{F}^{p,*}_c(U)$ of compactly supported sections is also well defined. Observe that on (m,q) forms the $d = d'$, since $\Lambda^{p,q} \simeq \Lambda^{p,0} \otimes \Lambda^{0,q}$, it is convenient to consider just the case $p = m$.

Example 1. *Let T' be the locally integrable structure the real-complex on the Euclidean space $\mathcal{M} = \mathbb{C}^{\nu}_z \times \mathbb{R}^d_s \times \mathbb{R}^l_t$ spanned by the complex differentials $dz_1, \ldots, dz_{v'}$, $0 \le v' \le v$, and the real differentials $ds_1, \ldots, ds_d$, and $dt_1, \ldots, dt_{l'}, 0 \le l' \le l$. The characteristic set T^0 is spanned by differentials ds_i's and dt_j's. In general, the structure T' is not necessarily a structure of any of the types (i)-(iv) defined above. When $v = d = 0$, we T' is the partial de Rham structure on the t-space $\mathbb{R}^l$, and the de Rham structure $T' = 0$ if $l' = 0$ also. When $l = 0$ and $v' = v$, we get an example of a flat CR (Cauchy-Riemann structure) on $\mathcal{M}$, which is a generalization of the Dolbeault $\overline{\partial}$-structure $(d = 0)$ on the z-space $\mathbb{C}^{\nu}$.*

The prototype of locally integrable structure our purposes is when T' is the cotangent bundle of a real-complex submanifold of $\mathcal{M} = \mathbb{C}^{\nu}_z \times \mathbb{R}^d_s \times \mathbb{R}^l_t$. Alternatively, we can describe this structure as follows.

Example 2. *Let T' be a locally integrable structure of rank $m = \nu + d$ in a neighbourhood $\mathcal{U}$ of the origin in $\mathcal{M} := \mathbb{C}^{\nu}_z \times \mathbb{R}^d_s \times \mathbb{R}^l_t$ defined by the exact differentials*

$$dz_1, dz_2, ..., dz_\nu, dw_1, dw_2, ..., dw_d, \tag{1.1}$$

$$w = s_k + \imath\varphi_k(z,\overline{z},s,t),\ \varphi_k(0) = 0, d\varphi_k(0) = 0,\ 1 \le k \le d. \tag{1.2}$$

Here to emphasize the complex nature of the variables $z = (z_1, z_2, ..., z_\nu)$, we have written φ_k as a function of z and $\overline{z}$ treated as independent variables rather than their real and imaginary parts x and y, respectively. The characteristic set T^0_o at the origin is spanned by differentials $ds_1, ds_2, ..., ds_d$. This structure is elliptic if and only if $d = 0$ and it is CR if and only if $l = 0$. When $d = 1$, $\mathcal{V} = T'^{\perp}$ is spanned by

$$\begin{aligned} L_j &= \frac{\partial}{\partial \overline{z}_j} - \imath\frac{\varphi_{\overline{z}_j}}{1+\imath\varphi_s}\frac{\partial}{\partial s}, 1 \le j \le \nu, \\ L_{\nu+i} &= \frac{\partial}{\partial t_i} - \imath\frac{\varphi_{t_i}}{1+\imath\varphi_s}\frac{\partial}{\partial s}, 1 \le i \le l. \end{aligned} \tag{1.3}$$

The local representation of d' is given by

$$d'f = \sum_{j=1}^{\nu} L_j f \cdot d\overline{z}_j + \sum_{j=1}^{l} L_{\nu+i} f \cdot dt_i.$$

Unlike the structures in Example 1, the characteristic set T^0 here is not necessarily a bundle. For instance, take $d = 1$ and $\varphi := t^2$, the fibre of T^0 over any point $t = 0$ is spanned by ds but $T^0 = 0$ otherwise.

2. Smoothing of Cohomology

For a pair (U', U), $U' \subseteq U$, of open sets in M, and $p = 0, 1, 2, ...$, there is a natural associated exact sequence

$$\cdots \to H^{p,q}(\mathcal{F}, \text{inc}) \to H^{p,q}(\mathcal{F}, U) \to H^{p,q}(\mathcal{F}, U') \to H^{p,q+1}(\mathcal{F}, U, U') \to \cdots$$

where inc : $U' \hookrightarrow U$ denotes the inclusion map. The cohomology of $H^{p,*}(\mathcal{F}, \text{inc})$ is the relative cohomology of $\mathcal{F}^{p,*}(U)$ with respect to the pair (U', U). We will use the notation $H^{p,*}(\mathcal{F}, U, U') := H^{p,*}(\mathcal{F}, \text{inc})$.

Let $\mathcal{F}$ be a function space such that there is an injection

$$\theta : C^\infty(U) \hookrightarrow \mathcal{F}(U).$$

Then there is a well-defined morphism

$$H^{p,q}(C^\infty; U, U') \stackrel{\theta_{p,q}}{\to} H^{p,q}(\mathcal{F}; U, U'),\ 0 \le p \le m, 0 \le q \le n,$$

$$[\alpha] \to [\theta(\alpha)].$$

The cohomology $H^{p,*}(\mathcal{F}, U, U')$ is said to be smoothable if $\theta_{p,q}$ is an isomorphism. Similarly, $H_c^{p,*}(\mathcal{F}, U, U')$ is smoothable if the restriction of $\theta_{p,q}$ to $\mathcal{F}_c^{p,*}(U)$ is isomorphism. For example, the de Rham theorem states that $\theta_{p,q}$ is an isomorphism (see [6]) for the de Rham-Dolbeault structure described in Example 1. In the case of elliptic structures T', there exist of homotopy formulas (see Section VI.7 in [8]) for the boundary operator $D = \overline{\partial}_z + d_t$, and thus the cohomology is always smoothable. In [1] it is proved that in the Maire's structure on $\mathbb{R}^4$, $\theta_{0,1}$ is not an isomorphism. Proposition 2 below will provide an abundance of examples of differential complexes with nonsmoothable cohomologies.

The surjectivity of $\theta_{p,q}$ is equivalent to the property that every cocycle in U on is cohomologous to a smooth form in U';

$(S^{p,q}) : \forall$ *cocycle* $f \in C^{(-\infty)}(U, \Lambda^{p,q}),\ \exists \alpha \in C^\infty(U', \Lambda^{p,q})$ *and*

$\beta \in C^{(-\infty)}(U', \Lambda^{p,q-1})$ *such that* $f = \alpha + d'\beta$ *in* U'.

On the other hand, the injectivity of $\theta_{p,q}$ is equivalent to the property that a smooth form that bounds a current also bounds a smooth form;

$(I^{p,q})$

$\forall\ \gamma = d'f \in C^\infty(U, \Lambda^{p,q})\ \exists \beta \in C^\infty(U', \Lambda^{p,q-1})$ *such that* $\gamma = d'f = d'\beta$ *in* U'.

If α in $S^{p,q}$ can always be chosen to be 0 form, then $S^{p,q}$ reduces to the property of local solvability of operator d', The properties $S^{p,q}$ and $I^{p,q+1}$ together can be viewed as the hypoellipticity, in the sense of F. Treves, of the operator d';

$(H_{p,q}) : \forall\ f \in C^{(-\infty)}(U, \Lambda^{p,q})$ *such that* $d'f \in C^\infty(U, \Lambda^{p,q+1})\ \exists \alpha \in C^\infty(U', \Lambda^{p,q})$

and $\beta \in C^{(-\infty)}(U', \Lambda^{p,q-1})$ *such that* $f = \alpha + d'\beta$ *in* U'.

Indeed, the 'only if' part is obvious, and to show the converse suppose $f \in C^{(-\infty)}(U, \Lambda^{p,q})$ is such that $d'f \in C^\infty(U, \Lambda^{p,q+1})$. By $I^{p,q+1}$, there exists a $\beta \in C^\infty(U', \Lambda^{p,q})$ such that $d'f = d'\beta$ in U'. By applying $S^{p,q}(U', U)$ to the cocycle $f - \beta$, we get an $\alpha \in C^\infty(U', \Lambda^{p,q})$ and $\beta_1 \in C^\infty(U', \Lambda^{p,q-1})$ such that $f - \beta = \alpha + d'\beta_1$ in U'.

We will denote by $S_c^{p,q}$ and $I_c^{p,q}$ the statements obtained by replacing $C^{(-\infty)}$ and C^∞ by $C_c^{(-\infty)}$ and C_c^∞ in $S^{p,q}$ and $I^{p,q}$, respectively .

Let $T^{'}$ be an elliptic structure on an open subset V of $\mathbb{C}_z^\nu \times \mathbb{R}_t^l$. Let $(z, \overline{z}, t)$ be local coordinates in V such that $T^{'}\Big|_V$ generated by $dz_1, \ldots, dz_\nu$ where $z_j = x_j + \imath y_j,\ 1 \le j \le \nu$. The boundary operator for differential complex associated to $T^{'}\Big|_V$ is

$$D = \overline{\partial}_z + d_t.$$

If $\alpha \in C^\infty(V, \Lambda^{p,q})$ and $\beta \in C^\infty(V, \Lambda^{\nu-p,\mu-q})$ are such that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$ is relatively compact in V, define

$$< \alpha, \beta >= \int_V \alpha \wedge \beta.$$

If f_ε and u_δ denote regularizations (see [6]) of f and u, respectively, let

$$< f, u >= \lim_{\varepsilon,\delta\to 0} \int_V f_\varepsilon \wedge \beta_\delta.$$

The singular support $\mathrm{sing\,.\,supp}\, u$ of a distribution u is the set of all points x such that u is not C^∞ in any neighbourhood of x.The following lemma is well known (see [5]).

Lemma 1. *If* $\mathrm{sing\,.\,supp}\, u \cap \mathrm{sing\,.\,supp}\, Df = \mathrm{sing\,.\,supp}\, Du \cap \mathrm{sing\,.\,supp}\, f = \emptyset$,*and* $\mathrm{supp}\, f$ *or* $\mathrm{supp}\, u$ *is compact, then* $< f, u >$ *well defined.*

3. A Priori Estimates

Proposition 1. *Let* $U' \subset U$ *be a a open sets in* $\mathcal{M}$.

(i) If $S^{p,q}(U', U)$, $0 \le p \le m, 1 \le q \le n$, *holds, then* $\forall\ K' \subset\subset U'$, $\forall k, \nu \in \mathbb{Z}_+$, $\exists$ $K \subset\subset U$, $l \in \mathbb{Z}_+$, *and a constant* $C = C(K', k, \nu) > 0$ *such that the inequality*

$$\left| \int f \wedge u \right| \le C(\|f\|_{k,K}) \left(\|u\|_{(-\nu)} + \|d'u\|_{l,K'} \right) \tag{3.1}$$

holds for all cocycles $f \in C^\infty(U, T'^{p,q})$ *and for all* $u \in C^\infty_{K'}(U', T'^{m-p,n-q})$;

(ii) If $S_c^{p,q}(U', U)$, $0 \le p \le m, 1 \le q \le n,$, *then* $\forall K \subset\subset U$, $\forall\ k, \nu \in \mathbb{Z}_+$, $\exists$ $K' \subset\subset U'$, $l \in \mathbb{Z}_+$, *and a constant* $C = C(K, k, \nu) > 0$ *such that the inequality*

$$\left| \int f \wedge u \right| \le C(\|f\|_{k,K}) \left(\|u\|_{K',(-\nu)} + \|d'u\|_{l,K'} \right) \tag{3.2}$$

holds for all cocycles $f \in C^\infty_K(U, T'^{p,q})$ *and for all* $u \in C^\infty(U', T'^{m-p,n-q})$.

Proof. Consider the linear space

$$F = \{f \in C^k(U, T'^{p,q}) : d'f = 0\} \tag{3.3}$$

with induced topology from $C^k(U, T'^{p,q})$ ($d'f$ here is to be understood in the sense of distributions if $k = 0$.) We claim that F is complete, and thus a Frechet space. Indeed, a Cauchy sequence $\{f_i\}$ in F has a limit $f \in C^k(U, T'^{p,q})$. For a compact set K and any $\psi \in C^\infty_K(U, T'^{m-p,n-q})$,

$$|d'f \wedge \psi| = |d(f - f_j) \wedge \psi| = \left| \int (f - f_j) \wedge d\psi \right| \le \|f - f_j\|_{0,K} \int_K |\psi| \to 0 \text{ as } j \to 0.$$

Thus $d'f = 0$ in the sense of distributions.

Let $E_{K'} = C^\infty_{K'}(U^{'}, T'^{m-p,n-q})$ but now endowed with topology defined by the seminorms $\|u\|_{(-\nu)} + \|d'u\|_l$ where $l = 0, 1, 2, ...$ On the product space $F \times E$, the bilinear form

$$F \times E \ni (f, u) \longrightarrow \int f \wedge u. \tag{3.4a}$$

is a linear continuous map on F when $u \in E$ is fixed because

$$\left| \int f \wedge u \right| \leq \|f\|_{0,K'} \cdot \int |u|. \tag{3.5}$$

Now fix $f \in F$, by $S^{p,q}(U', U)$ there is an $\alpha \in C^\infty(U', T'^{p,q})$ and a $\beta \in C^{(-\infty)}(U', T'^{p,q-1})$ such that $f = \alpha + d'\beta$ in U'. If we fix one such pair α and β, then there is a constant $C = C(K', \beta)$ such that

$$\left| \int d'\beta \wedge u \right| = \left| \int \beta \wedge d'u \right| \leq C\|d'u\|_{K',l}, \tag{3.6}$$

where l is the order of the current β on K'. (Here again $\int d'\beta \wedge u := (d'\beta, u)$ is to be understood in the sense of distributions.) Let $\chi \in C^\infty_0(U)$ be such that with $\chi = 1$ in a neighbourhood of K'. Now,

$$\left| \int \chi\alpha \wedge u \right| \leq \|u\|_{(-\nu)} \cdot \|\chi\alpha\|_{(\nu)}\,. \tag{3.7}$$

Thus, there exist $l \in \mathbb{Z}_+$ and constant $C_1 = C_1(f, K')$ such that

$$\left| \int f \wedge u \right| = \left| \int \chi\alpha \wedge u \right| + \left| \int d'\beta \wedge u \right| \tag{3.8}$$

$$\leq C_1(\|u\|_{(-\nu)} + \|d'u\|_l). \tag{3.9}$$

Thus the bilinear form (3.4a) is separately continuous in both variable. Since E is meterizable and F is a Frĕchet space, by the Banach-Stienhaus theorem it is jointly continuous. Thus the estimate (3.1) holds for some $C > 0$.

The estimate (3.2) follows by repeating the above argument with F replaced by $F_K = F \cap C^k_K(U, T'^{p,q})$ with norm C^k norm $\|\cdot\|_{K,k}$, and E by $C^\infty(U, T'^{m-p,n-q})$ with topology induced by the seminorms $\|u\|_{K',(-\nu)}$, $\|d'u\|_{K',l}$ where K' runs over an exhausting sequence of compact subsets K' of U' and l over nonnegative integers. ∎

Corollary 1. *If $S^{m,n}$ holds for a pair (U', U), then for every real numbers μ and ν there exists a constant $C_{\mu\nu}$ such that*

$$\|u\|_{(-\mu),K} \leq C_{\mu\nu} \left(\|u\|_{(-\nu)} + \|d'u\|_{l,K'}\right), \forall u \in C^\infty_{K'}(U'). \tag{3.10}$$

Proof. Consider the subspace

$$F = \{f \in H_{(\mu)}(U, T'^{p,q}) : d'f = 0\} \tag{3.11}$$

of $H_{(-\mu)}(U, T'^{p,q})$. If $f_j \to f, f_j \in F$, then for all $\psi \in C^\infty_0(U, T'^{m-p,n-q})$

$$|d'f \wedge \psi| = |d(f - f_j) \wedge \psi| = \left| \int (f - f_j) \wedge d\psi \right| \leq \|f - f_j\|_{(\mu)} \;\; \|\psi\|_{(-\mu)} \to 0 \text{ as } j \to 0.$$

Thus F is closed. By repeating the above argument we get

$$\left| \int f \wedge u \right| \leq C(\|f\|_{(\mu)}) \left(\|u\|_{(-\nu)} + \|d'u\|_{l,K'}\right), \forall f \in H_{(\mu)}(U, T'^{p,q}),$$

which implies the desired estimate. ∎

4. The Structures with nondegenerate Levi Form

Let $\mathcal{M} := \mathbb{C}^\nu_z \times \mathbb{R}^r_s \times \mathbb{R}^l_t \simeq \mathbb{C}^{\nu'}_{z'} \times \mathbb{C}^{\nu-\nu'}_{z''} \times \mathbb{R}^r_s \times \mathbb{R}^{l'}_{t'} \times \mathbb{R}^{l-l'}_{t''}$, $0 \le \nu' \le \nu, 0 \le l' \le l$, where

$$\begin{aligned} z' &= (z_1, z_2, \dots z_{\nu'}),\ t' = (t_1, t_2, \dots t'_{l'}); \\ z'' &= (z_{\nu'+1}, \dots z_\nu),\ t'' = (t_{l'+1}, t_2, \dots t_l); \\ s &= (s_1, s_2, \dots s_r). \end{aligned} \tag{4.1}$$

Let T' be a locally integrable structure on an open neighbourhood $\mathcal{U}$ of the origin in $\mathcal{M}$ spanned by the exact differentials given in (1.1), and let corresponding vector fields $L_j, 1 \le j \le \nu + r$, be as in (1.3).

Proposition 2. *With notations as above, suppose further that there exist functions $h_i : \mathcal{U} \to \mathbb{C}$ (partial solutions), $i = 1, 2$, with the following properties*

$$L_j h_1 = 0,\ \forall j \in \{1, 2, ..., \nu', \nu + 1, ..., \nu + l'\}, \tag{4.2}$$

$$\begin{aligned} L_j h_2 &= 0,\ \forall j \in \{\nu' + 1, ..., \nu, \nu + l' + 1, ..., \nu + l\}, \\ h_i(0) &= 0,\ d\,\mathrm{Im}\, h_i(0) = 0,\ d\,\mathrm{Re}\, h_i \neq 0, i = 1, 2, \end{aligned} \tag{4.3}$$

and for all $\forall (z, s, t) \in \mathcal{U}$,

$$\begin{aligned} \mathrm{Im}\, h_1 &\ge a\left(|z'|^2 + |s|^2 + |t'|^2\right) \\ \textit{and}\ \mathrm{Im}\, h_2 &\ge a\left(|z''|^2 + |s|^2 + |t''|^2\right), \end{aligned}$$

for some constant $a > 0$. Then $S^{0,q}(U', U)$ and $S^{0,n-q}(U', U)$, $m = \nu + r, q = \nu' + l' \ge 1$, $n = \nu + l$, can not hold for any sufficiently small pair (U', U), $0 \in U' \subseteq U$, of open subsets in $\mathcal{U}$.

Proof. From the local representation of d' and (4.2), it follows that

$$\begin{aligned} d'h_1 \wedge d\overline{z''} \wedge dt'' &= 0, \\ d'h_2 \wedge d\overline{z'} \wedge dt' &= 0. \end{aligned} \tag{4.4}$$

Let U', U, $0 \in U' \subset U$, be a pair of sufficiently small open subsets of $\mathcal{U}$. Because of (4.4) then the form

$$f_\rho := e^{\iota\rho h_1} d\overline{z''} \wedge dt'' \in C^\infty(U, T'^{0,\nu+l-\nu'-l'}).$$

is d'-closed and thus represents a cocycle.

For a compact neighbourhood $K \subset\subset U$ of U' and any integer $k \ge 0$, there is a constant $C_{K,k} > 0$ such that

$$\|f_\rho\|_{K,k} \le C_{K,k}\rho^k, \forall \rho > 1.$$

Choose $\varepsilon > 0$ sufficiently small so that

$$B = \left\{(z, s, t) : |z'|^2 + |t'|^2 \le \varepsilon,\ |z''|^2 + |s|^2 + |t''|^2 \le \varepsilon\right\} \subset\subset U'.$$

Let $\chi \in C^\infty_c(U')$, $0 \le \chi \le 1$, such that $\chi = 1$ on B. If we define

$$u_\rho := e^{\iota\rho h_2} \chi d\overline{z'} \wedge dt' \wedge dz \wedge dw \in C^\infty(U', T'^{\nu+d,\nu'+l'}),\ \forall \rho > 1.$$

then

$$d'u_\rho = e^{\iota\rho h_2}\,(d\chi) \wedge d\overline{z'} \wedge dt' \wedge dz \wedge dw.$$

Since $\operatorname{Im} h_2 \geq a\varepsilon$ on B, we have any integer $l \geq 0$

$$\|d' u_\rho\|_l \leq C_l \rho^l e^{-a\varepsilon\rho}, \forall \rho > 1$$

for some constant $C_l > 0$. Because $\cup_\rho \operatorname{supp} u_\rho \subset \operatorname{supp} \chi$, by Lemma 26.4.15 in [7], for any $\mu \in \mathbb{R}$, there exists a constant C_μ such that

$$\|u_\rho\|_{(-\mu)} \leq C_\mu \rho^{-\mu}, \forall \rho > 1. \tag{4.5}$$

The conditions on h_1 and h_2 yield the lower bound

$$\rho \operatorname{Im}(h_1 + h_2)\left(\frac{z}{\sqrt{\rho}}, \frac{\overline{z}}{\sqrt{\rho}}, \frac{s}{\sqrt{\rho}}, \frac{t}{\sqrt{\rho}}\right) \geq a\varepsilon, \forall (z, \overline{z}, s, t) \in U.$$

Thus

$$\left| e^{\imath\rho(h_1+h_2)} \chi \det(I_r + \imath\varphi_s) \right| \leq C e^{-a\left(|z|^2+2|s|^2+|t|^2\right)} \text{ on } U,$$

where I_r denotes the $r \times r$ identity matrix and $C = \sup_U |\det(I_r + \imath\varphi_s)|$.

By the Lebesgue dominated convergence theorem we have

$$\rho^{\frac{m+n}{2}} \int f_\rho \wedge u_\rho = \rho^{\frac{m+n}{2}} \int e^{\imath\rho(h_1+h_2)} \chi \det(I_r + \imath\varphi_s) dxdydsdt$$

$$= \int e^{\imath\rho(h_1+h_2)\left(\zeta/\sqrt{\rho}\right)} \chi \det(I_r + \imath\varphi_s)\left(\frac{z}{\sqrt{\rho}}, \frac{\overline{z}}{\sqrt{\rho}}, \frac{s}{\sqrt{\rho}}, \frac{t}{\sqrt{\rho}}\right) dxdydsdt$$

$$\to C_1 \int e^{-(a_1+a_2)(|z|^2+|s|^2+|t|^2)} \, dxdydsdt \neq 0,$$

for some constant $C_1 \neq 0$.

Now taking the estimate (4.5) into account we see that if $\mu >> m + n + k$, no constant C can satisfy the inequality

$$\begin{aligned} \rho^{\frac{m+n}{2}} \int f_\rho \wedge u_\rho &\leq C\rho^{\frac{m+n}{2}} \|f_\rho\|_{K,k} \left(\|u_\rho\|_{(-\mu)} + \|d' u_\rho\|_{K,l}\right) \\ &\leq C\rho^{k+\frac{m+n}{2}} \left(\rho^{-\mu+\frac{m+n}{2}} + \rho^l e^{-a_2 b\rho}\right) \end{aligned}$$

for all $\rho > 1$. Thus the $S^{0,q}$ does not hold for the pair (U', U). By interchanging the role of variables (z', t') and (z'', t''), we conclude that $S^{0,n-q}$ does not hold either for (U', U). ∎

Corollary 2. *Let $\varphi' : \mathbb{C}^{\nu'} \times \mathbb{R}^{l'} \to \mathbb{R}$ and $\varphi'' : \mathbb{C}^{\nu-\nu'} \times \mathbb{R}^{l-l'} \to \mathbb{R}$ be two C^∞ functions such that*

$$\varphi'(z', t') \geq a\left(|z'|^2 + |t'|^2\right) \text{ and } \varphi''(z'', t'') \geq a\left(|z''|^2 + |t''|^2\right), \ \forall (z, t) \in \overline{U}$$

for some constant $a > 0$. Let T' be the locally integrable structure on $\mathbb{C}^\nu_z \times \mathbb{R}_s \times \mathbb{R}^l_t$ defined by $dz_1, dz_2, ..., dz_\nu$, and dw, where

$$w := s + \imath(\varphi'(z', t') - \varphi''(z'', t'')).$$

Then $S^{0,q}(U', U)$ and $S^{0,n-q}(U', U)$ does not hold any pair of (U', U) open sets.

Proof. The functions $h_1 := s + \imath\varphi'(z', t')$ and $h_2 := s + \imath\varphi''(z'', t'')$ satisfy all the hypothesis of Proposition 2. ∎

Now suppose there exists a $\sigma \in \mathbb{R}^r$ such that the Levi form associated with the locally integrable structure T' at the characteristic vector $(0, \sigma \cdot ds)$ has $q \geq 1$ positive eigenvalues and $n - q = \nu + l - q$, negative eigenvalues and its restriction to $\mathcal{V}_0 \cup \overline{\mathcal{V}}_0$ is nondegenerate. By our assumptions on $\varphi_k' s$ at the origin the vector fields $L_j, 1 \leq j \leq \nu$, and $L_{\nu+i}, 1 \leq i \leq l$ reduce to $\frac{\partial}{\partial \bar{z}_j}$, $1 \leq j \leq \nu$ and $\frac{\partial}{\partial t_i}, 1 \leq i \leq l$, respectively. For $(0, \sigma \cdot ds) \in T_0^0$ we write

$$\Phi = \sigma_1 ds_1 + \sigma_2 ds_2 + ... + \sigma_r ds_r.$$

Under the hypothesis on the Levi form, after linear change of coordinates, if necessary, we can write the Levi form at $(0, \sigma \cdot ds)$ as follows

$$\Phi(z, \overline{z}, 0, t) = |z'|^2 + |t'|^2 - |z''|^2 - |t''|^2$$

$$+\text{terms involving } \operatorname{Re}(z_i t_j) \text{ and } \operatorname{Re}(z_i z_j)) \text{ and terms of higher order.}$$

Lemma 2. *(Theorem VIII.3.1 [8]) There exist sufficiently small $\varepsilon \geq 0$ and $\varepsilon' \geq 0$ and a solution of the form*

$$h = w_r + \iota \varepsilon' \sum_{k=1}^{r} w_k^2$$

such that

(4.6)
$$\frac{1}{2}(|z'|^2+\varepsilon'|s|^2+|t'|^2)-\frac{3}{2}(|z''|^2+|t''|^2) \leq \operatorname{Im} h \leq \frac{3}{2}(|z'|^2+\varepsilon|s|^2+|t'|^2)-\frac{1}{2}(|z''|^2+|t''|^2),$$

for all $(z, s, t) \in U$

Now that the functions defined as

$$\begin{aligned} h_1 &= -h + 4\iota \left(|z'|^2 + |t'|^2\right) + 4\iota \sum_{k=1}^{r} w_k^2, \\ h_2 &= h + \iota 4 \left(|z''|^2 + |t''|^2\right) \end{aligned}$$

satisfy the hypothesis of Theorem 2 follows from the above lemma.

We have proved

Theorem 1. *Suppose the Levi form of the locally integrable structure T' at the characteristic vector $(0, \sigma \cdot ds)$ has $q \geq 1$ positive eigenvalues and $n - q = \nu + l - q$ negative eigenvalues, and its restriction to $\mathcal{V}_0 \cup \overline{\mathcal{V}}_0$ is nondegenerate. Then for any sufficiently small neighbourhoods $U' \subset U$ of the origin, and for any integer $k \geq 0$, there exists a a cocycle $f \in C^k(U, \Lambda^{0,q})$ that is* not *cohomologous to any smooth current on U'. In particular, $S^{0,q}(U', U)$ does not hold.*

5. Smoothing at the Top Degree

When $p = m, q = n$, $S^{m,n}$ reduces to the solvability modulo C^∞ :

$\forall\ f \in C^{(-\infty)}(U, \Lambda^{m,n})$, $\exists \alpha \in C^\infty(U', \Lambda^{m,n})$ and $\beta \in C^{(-\infty)}(U', \Lambda^{m,n-1})$ *such that* $f = \alpha + d'\beta$ *on* U'.

Theorem 2. *Let $\mathcal{U}$ be an open set in M. If there is a C^∞ solution $h : \mathcal{U} \to \mathbb{C}$ and a set $K_0 \subset\subset \mathcal{U}$ such that*

(5.1) $\quad \operatorname{Im} h = 0$ *on* K_0, $\operatorname{Im} h > 0$ *on* $\mathcal{U} \setminus K_0$ *and* $d \operatorname{Re} h \neq 0$ *on* $\mathcal{U}$,

then $S^{m,n}$ can not hold for any pair (U', U) of open sets with $K_0 \subset U' \subset U \subset \mathcal{U}$.

Proof. Let (U', U) be a pair of open sets such that $K_0 \subset U' \subset U \subset \mathcal{U}$. Choose a compact set $K_1 \subset\subset U'$ that contains K_0 in its interior. For a $\psi \in C_0^\infty(U')$ such that $\psi = 1$ on K_1, put

$$v_\rho = e^{\imath\rho h}\psi.$$

Since h is a solution, $d'v_\rho = e^{\imath\rho h} d'\psi$ and

$$\|d'v_\rho\|_l = \left\|e^{\imath\rho h} d'\psi\right\|_l \leq \text{const}\,.\rho^l e^{-\varepsilon\rho},\ \varepsilon := \max_{K_1} \operatorname{Im} h.$$

Since $\operatorname{Im} h(0) = 0$ and $\psi(0) \neq 0$, by Lemma 26.4.15 in [7] there exists a constant $C > 1$ such that

$$C^{-1}\rho^{-\frac{m+n}{2}+\mu} \leq \|v_\rho\|_{(-\mu),K} \leq C\rho^{\frac{m+n}{2}-\mu}, \forall\mu.$$

By the estimate 3.10, there exist a constant $A_{\mu\nu} > 0$ such that

$$\begin{aligned} \rho^{-\frac{m+n}{2}+\mu} &\leq \|v_\rho\|_{K,(-\mu)} \\ &\leq A_{\mu\nu}\left(\|v_\rho\|_{K,(-\nu)} + \|d'v_\rho\|_{K,l}\right), \end{aligned}$$

which implies

$$C^{-1} \leq A_{\mu,\nu}\left(C\rho^{m+n+\mu-\nu} + e^{-\varepsilon\rho}\rho^{-\nu}\right)$$

can not hold for all $\rho > 1$ for $\nu >> \mu + m + n$. ∎

6. Rigid structure on a hypersurface

Let $(\mathcal{M}, T^{'})$ be a locally integrable structure with its characteristic set T_o^0 at the origin of rank $r = 1$.

Let $\mathcal{U}$ be an open set in $\mathcal{M}$, and let (U', U), $U' \subseteq U \subseteq \mathcal{U}$ be a pair of open subsets.

We are going to make a number of hypotheses.

Hypothesis 1. *There exists a smooth solution (i.e. 0-cocycle)* $w : \mathcal{U} \to \mathbb{C}$ *with*

$$d(\operatorname{Re} w)(X) \neq 0, \forall X \in \mathcal{U} \text{ and} \tag{6.1}$$

and an elliptic structure $T_{\#}^{'}$ *on* $\mathcal{U}$ *such that*

$$T'|_{\mathcal{U}} = T_{\#}^{'} \oplus \mathbb{C}dw \text{ and } d(\operatorname{Re} w) \notin T_{\#}^{'} \oplus \overline{T}_{\#}^{'}. \tag{6.2}$$

If (6.1) holds, then locally we may treat $s = \operatorname{Re} w$ as an independent variable, and coordinatize a local chart in $\mathcal{U}$ by the variables $(z, \overline{z}, s, t) \in \mathbb{C}_z^\nu \times \mathbb{R}_s \times \mathbb{R}_t^l$. In this chart, where T' is spanned by the exact differentials $dz_1, dz_2, ..., dz_\nu, dw$, $w = s + \imath\varphi(z, \overline{z}, s, t)$, and $T_{\#}^{'}$ by $dz_1, dz_2, ..., dz_\nu$. The level set

$$\Sigma = \{X \in \mathcal{U}; \operatorname{Re} w(X) = 0\}$$

then is a smooth oriented real hypersurface in $\mathcal{M}$. The orientation of Σ can determined by the one form $d(\operatorname{Re} w)$. Fix one such orientation and set

$$S = \{\xi \in \Sigma; \operatorname{Im} w \circ \pi^{-1}(\xi) = 0\} \simeq \{X \in \mathcal{U} : w(X) = 0\}. \tag{6.3}$$

There exists a smooth vector field ϑ defined on U such that

$$\vartheta(\zeta) = \vartheta(\overline{\zeta}) = 0, \forall\zeta \in T_{\#}^{'} \text{ and } \vartheta(\operatorname{Re} w) = 1. \tag{6.4}$$

Hypothesis 2 *The structure* T' *is rigid in the sense that*

$$\vartheta(\operatorname{Im} w)(X) = 0, \forall X \in \mathcal{U}. \tag{6.5}$$

For a rigid structure, the solution w in (6.1) may be chosen so that its imaginary part $\operatorname{Im} w = \varphi$ is independent of its real part $s = \operatorname{Re} w$. We refer the reader to Section VI.9 of [8] for the Lie algebraic characterizations and other properties of rigid structures.

Let $\psi \in C^\infty(\Sigma)$ a sufficiently small positive function on Σ.

For each $\xi \in \Sigma$, let γ_ξ be the segment of the of integral curve of ϑ centered at $\xi \in \Sigma$ defined by the inequality $|\operatorname{Re} w| < \psi(\xi)$ such that

$$\overline{\gamma}_\xi \cap \overline{\gamma}_{\xi'} = \emptyset, \forall\, \xi, \xi^{'} \in \Sigma, \xi \neq \xi^{'}, \tag{6.6}$$

where $\overline{\gamma}_\xi$ denotes the closure of γ_ξ. Let

$$\Gamma = \bigcup_{\xi \in \Sigma} \gamma_\xi. \tag{6.7}$$

The following conditions will ensure that the intersections $U \cap \Sigma$ and $U' \cap \Sigma$ are topologically simple enough.

Hypothesis 3. *The pair U and U' have the following properties.*

(i) $\Sigma \cap \overline{U'} \subset\subset U$,

(ii) $\gamma_\xi \cap (U \cap \Sigma) \neq \phi, \forall \xi \in U$,

(iii) The de Rham-Dolbeault cohomology $H^{0,*}(C^\infty, U \cap \Sigma)$ and the reduced homology $\widetilde{H}_{0,*}(U^{'} \cap \Sigma)$ both vanish.

This allows us to choose ψ sufficiently small so that

$$\overline{U' \cap \Gamma} \subset\subset U. \tag{6.8}$$

The map $\Sigma \hookrightarrow \Gamma$ given by $\xi \to \gamma_\xi(0)$ is an injection and we can view Γ as a fiber bundle over Σ with γ_ξ as the fiber over the point $\xi \in \Sigma$;

$$\pi : \Gamma \longrightarrow \Sigma,\ \pi(\gamma_\xi) = \xi. \tag{6.9}$$

The rigidity hypothesis implies that

$$\operatorname{Im} w\,(X) = \operatorname{Im} w\,(\pi\,(X))\,, \forall X \in U. \tag{6.10}$$

The map π induces the pull-back map on complexified cotangent bundles;

$$\pi^* : \mathbb{C}T^*\Sigma \longrightarrow \mathbb{C}T^*\Gamma. \tag{6.11}$$

A local trivalization of the bundle (6.9) is given as follows;

$$\Gamma \ni X \xrightarrow{\cong} (\pi(X), \operatorname{Re} w(X)) \in \{(\xi, s) \in \Sigma \times \mathbb{R};\ |s| < \psi(\xi)\}. \tag{6.12}$$

If $(\xi, s) \in \Gamma$ and $\varpi \in \mathbb{C}T^*_\xi\Sigma$, then

$$\varpi \neq 0 \Rightarrow \pi^*\varpi \neq 0 \text{ at } (\xi, s). \tag{6.13}$$

Since the flow of ϑ preserves the elliptic substructure $T^{'}_{\#}$ we get the isomorphism $T^{'}_{\#} \cong \pi^* T^{'}_{\Sigma}$ and $d^{'}\pi^* = \pi^* d^{'}_{\Sigma}$.

For the structure in Example 2, we may take $\vartheta = \frac{\partial}{\partial s}$, $\Sigma = \mathbb{C}^{m-1} \times \{0\} \times \mathbb{R}^l$, $\gamma_{(z,\bar{z},t)} : s \to (z, s+\imath\varphi(z,\bar{z},t), t)$, $\Gamma \cap U = \{(z,s,t) : |s| \leq \psi(z,0,t), (z,0,t) \in \Sigma \cap U\}$, and $d'_{\Sigma} = d_t + \partial_{\bar{z}}$.

Let $U_0 = \pi(U)$, $U'_0 = \pi(U')$. We introduce the following condition.

$\mathbf{I}^+_q(U', U)$: *There exist currents* $f_0 \in C_c^{(-\infty)}(U_0, T_{\Sigma}^{\prime 0,q-1})$ *and* $u_0 \in C^{(-\infty)}(U'_0, T_{\Sigma}^{\prime m-1,n-q})$ *such that following are satisfied.*

$$\begin{aligned} \operatorname{Im} w \circ \pi &\geq 0 \text{ on } \operatorname{supp} u_0, && (6.14)\\ \operatorname{Im} w \circ \pi &> 0 \text{ on } \operatorname{supp} d^{'}_{\Sigma} u_0, && (6.15)\\ \operatorname{Im} w \circ \pi &\leq 0 \text{ on } \operatorname{supp}\ d^{'}_{\Sigma} f_0,\\ &< f_0, d^{'}_{\Sigma} u_0 > \neq 0. \end{aligned}$$

By $\mathbf{I}^-_q(U', U)$ *we mean the above condition with roles of* U^+ *and* U^- *are interchanged.*

Since $(\Sigma, T^{'}_{\Sigma})$ is an elliptic structure, there exists $\tilde{f}_0 \in C_c^{\infty}(U, \Lambda_{\Sigma}^{0,q-1})$ and $g_0 \in C_c^{\infty}(U, \Lambda_{\Sigma}^{0,q-2})$ such that

$$f_0 = \tilde{f}_0 + d^{'}_{\Sigma} g_0.$$

Thus by replacing f_0 with $\tilde{f}_0$, if necessary, we may assume $f_0 \in C_c^{\infty}(U_0, \Lambda_{\Sigma}^{0,q-1})$.

When $\mathbf{I}^+_q(U', U)$ or $\mathbf{I}^-_q(U', U)$ holds then level set

$$S = \{\xi \in \Sigma;\ \operatorname{Im} w|_{\gamma_\xi} = 0\}$$

is necessarily a singular hypersurface in Σ.

Example 3. *Let* $\mathcal{M} = \mathbb{C}^2_z \times \mathbb{R}_s \subset \mathbb{C}^2_z \times \mathbb{C}_w$, $T'\ dz_1, dz_2, dw$, *where* $w = s + \imath\left(|z_1|^2 - |z_2|^2\right)$, $d^{'}_{\Sigma} = \overline{\partial}$. *Let* $\chi_1(z_1) \in C^{\infty}$ *such that* $\chi_1(z_1) = 1$ *for all* $|z_1| \leq 4$. *Here* $m = 3$, $q = 1$, $n = 2$.

$$\begin{aligned} u_0 &:= \delta_{z_2} \chi_1(z_1)\, d\bar{z}_2 dz_1 dz_2 \in C^{(-\infty)}(U^+, \Lambda^{2,1}).\\ \overline{\partial} u_0 &= \delta_{z_2} \frac{\partial \chi_1}{\partial \bar{z}_1}(z_1)\, d\bar{z}_1 d\bar{z}_2 dz_1 dz_2\\ \operatorname{Im} w &\geq 0 \textit{ on } \operatorname{supp} u_0,\\ \operatorname{Im} w &\geq 4 \textit{ on } \operatorname{supp} \overline{\partial} u_0 \subset \{(z_1, 0) : 4 \leq |z_1|\}. \end{aligned}$$

$$\begin{aligned} f_0(z) &:= \begin{cases} \frac{1}{z_1} \textit{ if } |z_1|^2 \leq |z_2|^2 \leq 1 \\ 0 \textit{ if otherwise} \end{cases} \in C_c^{(-\infty)}\left(U^-, \Lambda^{0,0}\right),\\ \overline{\partial} f_0 &= \begin{cases} \delta_{z_1} d\bar{z}_1 \textit{ if } |z_1|^2 \leq |z_2|^2 \leq 1 \\ 0 \textit{ if otherwise} \end{cases} \in C_c^{(-\infty)}\left(U^-, \Lambda^{0,1}\right),\\ \operatorname{supp} \overline{\partial} f_0 &\subset \left\{(0, z_2) : |z_2|^2 \leq 1\right\} \subset U^- \cap B_1(0). \end{aligned}$$

$$\operatorname{Im} w \leq 0 \textit{ on } \operatorname{supp} \overline{\partial} f_0 \subset U^- \subset \left\{|z_1|^2 \leq |z_2|^2 \leq 1\right\}.$$

Since $\operatorname{supp}\overline{\partial}f_0$ *is compact and* $\operatorname{sing\,supp}\overline{\partial}u_0 \cap \operatorname{sing\,supp} f_0 = \phi, \left\langle u_0, \overline{\partial}f_0 \right\rangle$ *is well defined and*

$$\left\langle u_0, \overline{\partial}f_0 \right\rangle = \int \delta_{z_2}\chi_1(z_1)\,\delta_{z_1} d\overline{z}_1 d\overline{z}_2 dz_1 dz_2 = \chi_1(0) \neq 0$$

The above example can be modified to construct examples with $w := s + \imath\left(|z'|^2 + |t'|^2 - |z''|^2 - |t''|^2\right)$, where $z', z'', t', t''$are multivariables, as a solution.

Theorem 3. *Let* T', w, $T'_{\#}$, U, U', *and* Σ *be as above and Hypothesis 1-3 are satisfied. Suppose further that either* $I_q^+(U', U)$ *or* $I_q^-(U', U)$ *is satisfied. Then for any integer* $k \geq 0$ *there exists a cocycle* $f \in C_c^k(U, \Lambda^{0,q})$ *which is not cohomologous to any smooth* $(0, q)$*-form on* U'*. In particular, the cohomology group* $H_c^{0,q}(C^k, U, U') \ncong H_c^{0,q}(C^\infty, U, U')$ *for all* $k \geq 0$*.*

Proof. Suppose $I_q^+(U', U)$ holds. Let $S^{\pm} = \left\{\xi \in \Sigma : \pm \operatorname{Im} w \circ \pi^{-1}(\xi) \geq 0\right\}$ denote the two sides of $S = \{\operatorname{Im} w = 0\}$ in Σ. Let $\Lambda_\Sigma^{p,q}$ be the vector bundle of (p,q)-forms corresponding to the elliptic structure T'_Σ on Σ and let d'_Σ be the boundary operator of the associated differential complexes.

Let $b > 0$ be a sufficiently small number such that

$$b < \min_{\xi \in \overline{U \cap \Sigma}} \psi(\xi),$$

and

$$K := \{X \in \Gamma\,; \pi(X) \in \operatorname{supp} f_0,\ |\operatorname{Re} w| \leq 2b\} \subset\subset U' \cap \Gamma, \tag{6.16}$$

which is possible due to (6.8).

Let $a > 0$ be such that

$$\operatorname{Im} w(X) > 2a,\ \forall X \in \pi^{-1}\left(\operatorname{supp} d'_\Sigma u_0\right),$$

which exists due to our hypothesis $\mathbf{I}_q^+(U', U)$.

Select a function $F \in C_0^\infty(\mathbb{C})$ with the property

$$\operatorname{supp} F \subset \mathcal{R} = \{s + \imath r \in \mathbb{C} : |s| \leq b, \frac{a}{2} \leq r \leq \frac{3a}{2}\},$$

and $F > 0$ in the interior of $\mathcal{R}$.

Define

$$F_\rho(X) = F(\rho w(X)) \text{ on } |s| < \frac{b}{\rho}, \pi(X) \in U_0,\ F_\rho = 0 \text{ outside.}$$

Since by the definition of Γ, $|\operatorname{Re} w| = \psi(\pi(X))$ on $\partial\Gamma$, we conclude that $F_\rho|_{U\cap\Sigma}$, for $\rho > 1$, vanishes identically in the neighborhood of $U \cap \partial\Gamma$.

For $\rho > 1$, let $\dot{f}_\rho \in C_c^{(-\infty)}(U, \Lambda^{0,q})$ be the current represented by

$$f_\rho := F_\rho d\overline{w} \wedge f_0.$$

Clearly

$$\cup_{\rho>1} \operatorname{supp}(f_\rho) \subset K.$$

Lemma 3. *The current form* $\dot{f}_\rho \in C_c^\infty(U, \Lambda^{0,q})$ *and* $d'\dot{f}_\rho = 0$ *for all* $\rho > 1$*.*

Proof. First we show that the representative f_ρ is itself smooth by showing that $\operatorname{sing\,supp} f_0 = \phi$. Since d'_Σ is an elliptic operator, $\operatorname{sing\,supp} f_0 = \operatorname{sing\,supp} d'_\Sigma f_0 \subseteq \operatorname{supp} d'_\Sigma f_0$. By $I_q^+(U', U)$ and by (6.10), we have

$$\operatorname{Im} w(X) = \operatorname{Im} w \circ \pi\,(X) \leq 0,\ \forall X\ \in \pi^{-1}\left(\operatorname{sing\,supp} f_0\right).$$

On the other hand, the definition of F_ρ implies

$$\operatorname{Im} w \circ \pi\,(X) \geq \frac{a}{2\rho} > 0, \forall X \in \operatorname{supp} F_\rho.$$

Since $F_\rho(w) \in C^\infty$,

$$\operatorname{sing\,supp} f_\rho \subseteq \operatorname{supp} F_\rho \cap \pi^{-1}\left(\operatorname{sing\,supp} f_0\right) = \phi,$$

and thus $f_\rho \in C_c^\infty(U, \Lambda^{0,q})$.

To show that f_ρ is a cocycle, we compute

$$d' f_\rho = d'\left(F_\rho \varpi\right) \wedge \pi^* f_0 - F_\rho \Lambda \pi^* d'_\Sigma f_0.$$

By repeating the above argument we conclude that

$$\operatorname{supp}(F_\rho \varpi \wedge \pi^*(d'_\Sigma f_0)) \subseteq \operatorname{supp} F_\rho \cap \pi^{-1}\left(\operatorname{supp} d'_\Sigma f_0\right) = \phi. \tag{6.17}$$

Since

$$d(F_\rho dw) \wedge \pi^* f_0 = \frac{\partial F_\rho}{\partial w} dw \Lambda d\overline{w} \Lambda f_0 \in C^\infty(U, T'^{1,q}),$$

the current represented by it is 0 as a section $\Lambda^{0,q} = T'^{0,q+1}/T'^{1,q}$. Thus $d'\dot{f}_\rho = 0$. ∎

Let $\{u_\rho\}$ be a *regularization* (see [5] or [6]) of the current u_0 i.e. $u_\rho \in C^\infty(V \cap \Sigma, \Lambda^{m-1,n-q})$ and $u_\rho \to u_0$ in the sense of distributions. If $\dot{v}_\rho$ denotes the section $\Lambda^{m,n-q}$ represented by the smooth form

$$v_\rho = dw \wedge \pi^* u_\rho \in C^\infty(V, \Lambda^{m,n-q}), n = m + l, \rho > 1 \tag{6.18}$$

then $d'\dot{v}_\rho$is represented by

$$d' v_\rho = dw \wedge \pi^* d'_\Sigma u_\rho. \tag{6.19}$$

From the definition of F, it follows that the function

$$\widehat{G_\rho}(s + \imath t) \quad : \quad = \int_{-\infty}^{t} F(\rho(s + \imath t)) \tag{6.20}$$

$$= \begin{cases} 0 \text{ if } t < \frac{a}{2\rho}) \\ B(\rho s) \in C^\infty \text{ if } t \geq \frac{3a}{2\rho}) \end{cases}, \tag{6.21}$$

is in $C_0^\infty(U)$, where $B(s) > 0$ if $|s| < b$ and 0 if $|s| > b$. Now define for $\rho > 1$,

$$G_\rho(X) = \begin{cases} \widehat{G_\rho}(\rho w(X)) \text{ if } |s| < b, \pi(X) \in U_0 \\ 0 \text{ otherwise} \end{cases}.$$

By our choice of a and b, we have for all $\rho > 1$,

$$G_\rho(X) = B(\rho \operatorname{Re} w(X)) \text{ if } \pi(X) \in \operatorname{supp} d'_\Sigma u_\rho.$$

and

$$\operatorname{Im} w \geq \frac{a}{2\rho} \text{ on } \operatorname{supp}(G_\rho).$$

The section $\pi^*(f_0 \wedge u_\rho)$ has a unique representative of the kind

$$\sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K\} \wedge dz \in C^\infty(V, T'^{m-1,n-1}), \tag{6.22}$$

where $dz = dz_1 \wedge \cdots \wedge dz_{m-1}$. The definition of G_ρ implies that the section $F_\rho d\overline{w} \wedge dw/2\imath$ has the representative

$$F(\rho\,(s+\imath\varphi))ds \wedge d\varphi = -d[G_\rho(s,\varphi)ds]. \tag{6.23}$$

Since $p = m$, the section

$$F_\rho \varpi \wedge dw \wedge \pi^*(f_0 \wedge u_\rho)/2\imath$$

of $\Lambda^{m,n} = \Lambda^{m+n}\left(\mathbb{C}T^*\mathcal{M}\right)$ over $\mathcal{U}$ is a represented by

$$\pm d[G_\rho(s,\varphi)ds] \wedge \sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K \wedge dz \tag{6.24}$$

$$\begin{aligned} = \;& \pm d\left[G_\rho(s,\varphi)ds \wedge \sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K \wedge dz\right] \\ & \mp G_\rho(s,\varphi)ds \wedge d\left[\sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K \wedge dz\right]. \end{aligned} \tag{6.25}$$

The sections $d'_\Sigma(f_0 \wedge u_\rho)$ and

$$\pm d\left\{\sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K\right\} \wedge dz, \tag{6.26}$$

both belong to $C^\infty(V_0, \Lambda^{m-1,n}_\Sigma)$, the sections of top degree. Since

$$\Lambda^{m-1,n}_\Sigma = T'^{m-1,n}_\Sigma = \Lambda^{m+n-1}\mathbb{C} \otimes T^*\Sigma$$

any two sections of top degree are equal in U' up to a multiplication by a function, there exists a function $g \in C^\infty(U)$ such that

$$d'_\Sigma(f_0 \wedge u_\rho) = g(z,\overline{z},t) \cdot d\left\{\sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K\right\} \wedge dz,$$

Observe that

$$\int_{U\cap\Gamma} G_\rho(s,\varphi)ds \wedge \sum_{|J|+|K|=n-1} A^{(\rho)}_{J,K}(z,\overline{z},t)d\overline{z}_J \wedge dt_K \wedge dz = 0 \tag{6.27}$$

because the support of the integrand is a compact subset of U.

By (6.23),

$$\begin{aligned} v_\rho \wedge f_\rho &= \pm F_\rho d\overline{w} \wedge dw \wedge \pi^*(f_0 \wedge u_\rho)/2\imath \\ &= -d[G_\rho(s,\varphi)ds] \wedge \pi^*(f_0 \wedge u_\rho)/2\imath. \end{aligned}$$

By applying Stokes' theorem and using 6.24 and 6.27, we have

$$\begin{aligned}\int v_\rho \wedge f_\rho &= \int F_\rho d\overline{w} \wedge dw \wedge \pi^*(f_0 \wedge u_\rho) \\ &= \pm \int_{U\cap\Gamma} G_\rho ds \wedge \pi^* d_\Sigma^{'}[f_0 \wedge u_\rho]. \end{aligned} \tag{6.28}$$

Since $\varphi = \varphi(z, \overline{z}, t) > 0$ on $\pi(\operatorname{supp} G_\rho)$ for all $\rho > 1$ and Im $w \leq 0$ on the $\operatorname{supp} \pi^* d_\Sigma^{'} f_0$ we see that $G_\rho \wedge \pi^* d_\Sigma^{'} f_0 = 0$ and thus

$$G_\rho ds \wedge \pi^* d_\Sigma^{'}[f_0 \wedge u_\rho] = \pm G_\rho ds \wedge f_0 \wedge \left(d_\Sigma^{'} u_\rho\right) \tag{6.29}$$

Since $B(s) > 0$ on $(-b, b)$ and $< f_0, d_\Sigma^{'} u_0 > \neq 0$, we have

$$\left|\rho \int v_\rho \wedge f_\rho\right| = \left|\rho \left[\int G_\rho ds\right] \cdot \int \pi^* f_0 \wedge \left(d_\Sigma^{'} u_\rho\right)\right| \tag{6.30}$$

$$= \left|\rho \int B(\rho s) ds \cdot \int_{\Sigma\cap U} f_0 \wedge \left(d_\Sigma^{'} u_\rho\right)\right| \tag{6.31}$$

$$= \left|\int_{-b}^{b} B(s) ds \cdot \int_{\Sigma\cap U} f_0 \wedge \left(d_\Sigma^{'} u_\rho\right)\right| \tag{6.32}$$

$$\to \left[\int_{-b}^{b} B(s) ds\right] \cdot \left|< f_0, d_\Sigma^{'} u_0 >\right| \neq 0, \text{ as } \rho \to \infty. \tag{6.33}$$

Now define

$$f_\rho^* := e^{-\imath\rho w} f_\rho, \text{ and } v_\rho^* := e^{\imath\rho w} v_\rho,$$

Then

$$\lim_{\rho\to\infty} \rho \int f_\rho^* \wedge v_\rho^* = \lim_{\rho\to\infty} \rho \int f_\rho \wedge v_\rho \neq 0$$

For any compact set $K \subset\subset U$ we have

$$\left\|f_\rho^*\right\|_{k,K} = \sup e^{\rho\varphi} |f_\rho| \leq \rho^k e^{3a/2} \sup |f_1|$$

Since u_ρ is a regularization of u_0,there exists a $\varepsilon > 0$ such that

$$\operatorname{Im} w(X) \geq 2a - \frac{\varepsilon}{\rho} \geq a, \forall X \in K \cap \pi^{-1}(\operatorname{supp} d_\Sigma' u_\rho), \forall \rho >> 1.$$

Thus

$$\left\|d' v_\rho^*\right\|_{K,l} \leq \text{const} \cdot \rho^l e^{-a\rho}$$

As before by Lemma 26.4.15 in [7], we have for any compact set $K' \subset\subset U'$ there is a constant $C > 0$ such that

$$\left\|v_\rho^*\right\|_{K',(-\nu)} \leq C \cdot \rho^{-\nu + \frac{m+n}{2}}, \forall \nu > 0.$$

For $\nu >> m + n + 2k + 2$ the inequality 3.2 can not hold for all $\rho > 1$.

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