SHARPENING ON MIRCEA'S INEQUALITY

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ABSTRACT. In this paper, by using one of Sheng-Li Chen's theorems, combining the method of mathematical analysis and nonlinear algebraic equation system, Mircea's Inequality involving the area, circumradius and inradius of the triangle is sharpened.

1. Introduction and Main Results

Let S be the area, R the circumradius, r the inradius and p the semi-perimeter of the triangle, respectively. The following laconic and beautiful inequality is so-called Mircea's inequality in [1]

$$R + \frac{r}{2} > \sqrt{S}. \tag{1.1}$$

In 1991, D. S. Mitrinović et al. [2] noted a Mircea-type inequality that is obtained by D. M. Milošević

$$R + \frac{r}{2} \geq \frac{5}{6}\sqrt[4]{3}\sqrt{S}. \tag{1.2}$$

In [4], L. Carliz and F. Leuenberger strengthened inequality (1.2) as follows(see also [3])

$$R + r \geq \sqrt[4]{3}\sqrt{S}, \tag{1.3}$$

since (1.3) can be written

$$R + \frac{r}{2} \geq \frac{5}{6}\sqrt[4]{3}\sqrt{S} + \frac{1}{6}(R - 2r), \tag{1.4}$$

and from the well-known Euler's inequality $R \geqslant 2r$.

The main purpose of this article is to give a generalization of inequalities (1.2) and (1.3) or (1.4).

Theorem 1.1. *If $k \leqslant k_0$, then for any triangle, we have*

$$R + \frac{r}{2} \geq \frac{5}{6}\sqrt[4]{3}\sqrt{S} + k(R - 2r), \tag{1.5}$$

where k_0 is the root on interval $(\frac{11}{20}, \frac{4}{7})$ of the equation

$$2304k^4 - 896k^3 - 2336k^2 - 856k + 1159 = 0. \tag{1.6}$$

The equality in (1.5) is valid if and only if the triangle is isosceles one and the value of ratio of its sides is $2 : x_0 : x_0$, where x_0 is the positive root of the following equation

$$x^4 + 28x^3 - 120x^2 + 80x - 16 = 0.$$

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According to Theorem 1.1, we easily find some remarks.

Remark 1.1. k_0 is the best constant which makes (1.5) holding, and $k_0 = 0.5660532114\cdots$.

Remark 1.2. The function

$$f(k) = R + \frac{r}{2} - \frac{5}{6}\sqrt[4]{3}\sqrt{S} - k(R - 2r)$$

is a monotone increasing function on $(-\infty, k_0]$.

Remark 1.3. For $k = \frac{1}{2}$ in (1.5), then the inequality

$$R + 3r \geq \frac{5}{3}\sqrt[4]{3}\sqrt{S}$$

holds.

Remark 1.4. $x_0 = 3.079485433\cdots$.

2. Some Lemmas

In order to prove Theorem 1.1, we require several lemmas.

Lemma 2.1. ([5, 6] and see also [12]) (i) *If the inequality $p \geq (>) f_1(R, r)$ holds for any isosceles triangle whose top angle is greater than or equal to $60°$, then the inequality $p \geq (>) f_1(R, r)$ holds for any triangle.*
(ii) *If the inequality $p \leq (<) f_1(R, r)$ holds for any isosceles triangle whose top angle is less than or equal to $60°$, then the inequality $p \leq (<) f_1(R, r)$ holds for any triangle.*

Lemma 2.2. ([7]) *Denote*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and

$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m.$$

If $a_0 \neq 0$ or $b_0 \neq 0$, then the polynomials $f(x)$ and $g(x)$ have the common roots if and only if

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{vmatrix} = 0,$$

where $R(f,g)$ is called Sylvester's resultant of $f(x)$ and $g(x)$.

Lemma 2.3. ([7, 8]) *For a given polynomial $f(x)$ with real coefficients*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

if the number of the sign changes of the revised sign list of its discriminant sequence

$$\{D_1(f), D_2(f), \cdots, D_n(f)\}$$

is v, then, the number of the pairs of distinct conjugate imaginary roots of $f(x)$ equals v. Furthermore, if the number of non-vanishing members of the revised sign list is l, then, the number of the distinct real roots of $f(x)$ equals $l-2v$.

3. The Proof of Theorem 1.1

Proof. It's not difficult to find that the form of the inequality (1.5) is equivalent to $p \le (<) f_1(R,r)$ with the known identity $S = rp$. From Lemma 2.1, we easily see that inequality (1.5) holds if and only if this triangle is an isosceles triangle whose top angle is less than or equal to $60°$.

Let $a = 2, b = c = x (x \ge 2)$, then (1.5) is equivalent to

$$\frac{x^2}{2\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{2(x+1)} \ge \frac{5}{6}\sqrt[4]{3(x^2-1)} + k\left(\frac{x^2}{2\sqrt{x^2-1}} - \frac{2\sqrt{x^2-1}}{x+1}\right), \tag{3.1}$$

or

$$x^2 + x - 1 \ge \frac{5}{3}\sqrt[4]{3(x^2-1)^3} + k(x-2)^2. \tag{3.2}$$

For $x = 2$, (3.2) holds obviously. If $x > 2$, then (3.2) is equivalent to

$$k \le \frac{x^2 + x - 1 - \frac{5}{3}\sqrt[4]{3(x^2-1)^3}}{(x-2)^2}. \tag{3.3}$$

Define a function

$$g(x) = \frac{x^2 + x - 1 - \frac{5}{3}\sqrt[4]{3(x^2-1)^3}}{(x-2)^2} (x > 2). \tag{3.4}$$

Calculating the derivative for $g(x)$, we get

$$g'(x) = \frac{5[\sqrt[4]{3}(x^2+6x-4) - 6x\sqrt[4]{x^2-1}]}{6(x-2)^3\sqrt[4]{x^2-1}}. \tag{3.5}$$

Let $g'(x) = 0$, we obtain

$$\sqrt[4]{3}(x^2+6x-4) - 6x\sqrt[4]{x^2-1} = 0. \tag{3.6}$$

It's easy to find the roots of equation (3.6) must be the roots of the following equation

$$(x^4 + 28x^3 - 120x^2 + 80x - 16)(x+2)(x-2)^3 = 0. \tag{3.7}$$

For the range of the roots of equation (3.6) is on $(2, +\infty)$, the roots of equation (3.6) must be the roots of the equation as follows

$$x^4 + 28x^3 - 120x^2 + 80x - 16 = 0. \tag{3.8}$$

And it shows that equation (3.8) have only one positive root $x_0 \in (2, +\infty)$. Then $x_0 = 3.079485433\cdots$, and

$$\begin{aligned} g(x)_{min} = g(x_0) &= \frac{x_0^2 + x_0 - 1 - \frac{5}{3}\sqrt[4]{3(x_0^2-1)^3}}{(x_0-2)^2} \\ &= 0.5660532114\cdots \in (\frac{11}{20}, \frac{4}{7}). \end{aligned} \tag{3.9}$$

Therefore, the maximum of k is $g(x_0)$.

Now we prove that $g(x_0)$ is the root of the equation

$$2304k^4 - 896k^3 - 2336k^2 - 856k + 1159 = 0.$$

Considering the nonlinear algebraic equation system as follows

$$\begin{cases} x_0^4 + 28x_0^3 - 120x_0^2 + 80x_0 - 16 = 0 \\ u_0^4 - 3(x_0^2 - 1)^3 = 0 \\ x_0^2 + x_0 - 1 - \frac{5}{3}u_0 - (x_0 - 2)^2 t = 0 \end{cases} . \tag{3.10}$$

Hence, we have that $g(x_0)$ is also the solution of the nonlinear algebraic equation system (3.10). Eliminating u_0 and x_0 ordinal by resultant(by Lemma 2.2), we get

$$p_1(t)p_2(t)p_3(t) = 0, \tag{3.11}$$

where

$$p_1(t) = 2304t^4 - 896t^3 - 2336t^2 - 856t + 1159,$$

$$p_2(t) = 2304t^4 - 46976t^3 + 51104t^2 - 35496t + 10939,$$

$$\begin{aligned} p_3(t) = {} & 1327104t^8 - 27574272t^7 + 270856192t^6 - 218763264t^5 - 111704320t^4 \\ & + 78507776t^3 + 170893152t^2 - 164410112t + 62195869. \end{aligned}$$

The revised sign list of the discriminant sequence of $p_2(t)$ is

$$[1, 1, -1, -1]. \tag{3.12}$$

The revised sign list of the discriminant sequence of $p_3(t)$ is

$$[1, -1, -1, -1, 1, -1, 1, 1]. \tag{3.13}$$

So the number of the sign changes of the revised sign list of (3.12) equals 1, then with Lemma 2.2, the equation $p_2(t) = 0$ has 2 distinct real roots. And by using the function "realroot()"[10, 11] in Maple 9.0, we can find that $p_2(t) = 0$ has 2 distinct real roots in the following intervals

$$[\frac{1}{2}, \frac{17}{32}], [\frac{77}{4}, \frac{617}{32}] \tag{3.14}$$

and no real root on interval $(\frac{11}{20}, \frac{4}{7})$.

And the number of the sign changes of the revised sign list of (3.13) equals 4, then from Lemma 2.3, the equation $p_3(t) = 0$ has 4 pairs distinct conjugate imaginary roots. It is to say that $p_3(t) = 0$ has no real root.

From (3.9), it is deduced easily that $g(x_0)$ is the root of equation $p_1(t) = 0$. Namely, $g(x_0)$ is the root of equation (1.6).

Further considering the proof above, we easily obtain the required result of the equality in (1.5).

Thus, the proof of Theorem 1.1 is completed. □

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