SHARPENING ON MIRCEA’S INEQUALITY

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Abstract. In this paper, by using one of Sheng-Li Chen’s theorems, combining the method of mathematical analysis and nonlinear algebraic equation system, Mircea’s Inequality involving the area, circumradius and inradius of the triangle is sharpened.

1. Introduction and Main Results

Let $S$ be the area, $R$ the circumradius, $r$ the inradius and $p$ the semi-perimeter of the triangle, respectively. The following laconic and beautiful inequality is so-called Mircea’s inequality in [1]

$$R + \frac{r}{2} > \sqrt{S}.$$ (1.1)

In 1991, D. S. Mitrinović et al. [2] noted a Mircea-type inequality that is obtained by D. M. Milošević

$$R + \frac{r}{2} \geq \frac{5}{6} \sqrt[3]{3\sqrt{S}}.$$ (1.2)

In [2], L. Carliz and F. Leuenberger strengthened inequality (1.2) as follows (see also [3])

$$R + r \geq \sqrt[3]{3\sqrt{S}},$$ (1.3)

since (1.3) can be written

$$R + \frac{r}{2} \geq \frac{5}{6} \sqrt[3]{3\sqrt{S}} + \frac{1}{6}(R - 2r),$$ (1.4)

and from the well-known Euler’s inequality $R \geq 2r$.

The main purpose of this article is to give a generalization of inequalities (1.2) and (1.3) or (1.4).

Theorem 1.1. If $k \leq k_0$, then for any triangle, we have

$$R + \frac{r}{2} \geq \frac{5}{6} \sqrt[3]{3\sqrt{S}} + k(R - 2r),$$ (1.5)

where $k_0$ is the root on interval $(\frac{11}{20}, \frac{4}{7})$ of the equation

$$2304k^4 - 896k^3 - 2336k^2 - 856k + 1159 = 0.$$ (1.6)

The equality in (1.5) is valid if and only if the triangle is isosceles one and the value of ratio of its sides is $2 : x_0 : x_0$, where $x_0$ is the positive root of the following equation

$$x^4 + 28x^3 - 120x^2 + 80x - 16 = 0.$$
According to Theorem [1.1] we easily find some remarks.

**Remark 1.1.** \( k_0 \) is the best constant which makes \([1.5]\) holding, and \( k_0 = 0.5660532114 \cdots \).

**Remark 1.2.** The function
\[ f(k) = R + \frac{r}{2} - \frac{5}{6} \sqrt{3} \sqrt{S} - k(R - 2r) \]
is a monotone increasing function on \((-\infty, k_0]\).

**Remark 1.3.** For \( k = \frac{1}{2} \) in \([1.5]\), then the inequality
\[ R + 3r \geq \frac{5}{3} \sqrt{3} \sqrt{S} \]
holds.

**Remark 1.4.** \( x_0 = 3.079485433 \cdots \).

2. SOME LEMMAS

In order to prove Theorem [1.1] we require several lemmas.

**Lemma 2.1.** ([5, 6] and see also [12]) (i) If the inequality \( p \geq (>)f_1(R, r) \) holds for any isosceles triangle whose top angle is greater than or equal to 60°, then the inequality \( p \geq (>)f_1(R, r) \) holds for any triangle.

(ii) If the inequality \( p \leq (<)f_1(R, r) \) holds for any isosceles triangle whose top angle is less than or equal to 60°, then the inequality \( p \leq (<)f_1(R, r) \) holds for any triangle.

**Lemma 2.2.** ([7]) Denote
\[ f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \]
and
\[ g(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_m. \]

If \( a_0 \neq 0 \) or \( b_0 \neq 0 \), then the polynomials \( f(x) \) and \( g(x) \) have the common roots if and only if
\[
R(f, g) = \begin{vmatrix}
    a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\
    0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & \cdots & a_0 & \cdots & \cdots & a_n & 0 \\
    b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\
    0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m
\end{vmatrix} = 0,
\]
where \( R(f, g) \) is called Sylvester’s resultant of \( f(x) \) and \( g(x) \).

**Lemma 2.3.** ([7, 8]) For a given polynomial \( f(x) \) with real coefficients
\[ f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \]
if the number of the sign changes of the revised sign list of its discriminant sequence
\[ \{D_1(f), D_2(f), \cdots, D_n(f)\} \]
is \( v \), then, the number of the pairs of distinct conjugate imaginary roots of \( f(x) \) equals \( v \). Furthermore, if the number of non-vanishing members of the revised sign list is \( l \), then, the number of the distinct real roots of \( f(x) \) equals \( l - 2v \).

3. The Proof of Theorem 1.1

Proof. It’s not difficult to find that the form of the inequality (1.5) is equivalent to \( p \leq (\cdot)f_1(R, r) \) with the known identity \( S = rp \). From Lemma 2.1, we easily see that inequality (1.5) holds if and only if this triangle is an isosceles triangle whose top angle is less than or equal to 60°.

Let \( a = 2, b = c = x(x \geq 2) \), then (1.5) is equivalent to

\[
\frac{x^2}{2\sqrt{x^2 - 1}} + \frac{\sqrt{x^2 - 1}}{2(x + 1)} \geq \frac{5}{6} \sqrt[3]{3(x^2 - 1)} + k \left( \frac{x^2}{2\sqrt{x^2 - 1}} - \frac{2\sqrt{x^2 - 1}}{x + 1} \right),
\]

or

\[
x^2 + x - 1 \geq \frac{5}{3} \sqrt[3]{3(x^2 - 1)^3} + k(x - 2)^2.
\]

For \( x = 2 \), (3.2) holds obviously. If \( x > 2 \), then (3.2) is equivalent to \( k \leq \frac{x^2 + x - 1 - \frac{5}{3} \sqrt[3]{3(x^2 - 1)^3}}{(x - 2)^2} \).

Define a function

\[
g(x) = \frac{x^2 + x - 1 - \frac{5}{3} \sqrt[3]{3(x^2 - 1)^3}}{(x - 2)^2} (x > 2).
\]

Calculating the derivative for \( g(x) \), we get

\[
g'(x) = \frac{5(\sqrt[3]{3(x^2 + 6x - 4)} - 6x \sqrt{x^2 - 1})}{6(x - 2)^3 \sqrt{x^2 - 1}}.
\]

Let \( g'(x) = 0 \), we obtain

\[
\sqrt[3]{3(x^2 + 6x - 4)} - 6x \sqrt{x^2 - 1} = 0.
\]

It’s easy to find the roots of equation (3.6) must be the roots of the following equation

\[
x^4 + 28x^3 - 120x^2 + 80x - 16)(x + 2)(x - 2)^3 = 0.
\]

For the range of the roots of equation (3.6) is on \( (2, +\infty) \), the roots of equation (3.6) must be the roots of the equation as follows

\[
x^4 + 28x^3 - 120x^2 + 80x - 16 = 0.
\]

And it shows that equation (3.8) have only one positive root \( x_0 \in (2, +\infty) \). Then \( x_0 = 3.079485433 \cdots \), and

\[
g(x)_{min} = g(x_0) = \frac{x_0^2 + x_0 - 1 - \frac{5}{3} \sqrt[3]{3(x_0^2 - 1)^3}}{(x_0 - 2)^2}
\]

\[
= 0.5660532114 \cdots \in \left( \frac{11}{20}, \frac{4}{7} \right).
\]

Therefore, the maximum of \( k \) is \( g(x_0) \).

Now we prove that \( g(x_0) \) is the root of the equation

\[
2304k^4 - 896k^3 - 2336k^2 - 856k + 1159 = 0.
\]
Considering the nonlinear algebraic equation system as follows

\[
\begin{align*}
\begin{cases}
    x_0^4 + 28x_0^3 - 120x_0^2 + 80x_0 - 16 = 0 \\
    u_0^4 - 3(x_0^2 - 1)^3 = 0 \\
    x_0^2 + x_0 - 1 - \frac{5}{3}u_0 - (x_0 - 2)^2t = 0
\end{cases}
\end{align*}
\]  
(3.10)

Hence, we have that \( g(x_0) \) is also the solution of the nonlinear algebraic equation system (3.10). Eliminating \( u_0 \) and \( x_0 \) ordinal by resultant (by Lemma 2.2), we get

\[
p_1(t)p_2(t)p_3(t) = 0,\tag{3.11}
\]

where

\[
p_1(t) = 2304t^4 - 896t^3 - 2336t^2 - 856t + 1159,
\]

\[
p_2(t) = 2304t^4 - 46976t^3 + 51104t^2 - 35496t + 10939,
\]

\[
p_3(t) = 1327104t^8 - 27574272t^7 + 270856192t^6 - 218763264t^5 - 111704320t^4 + 78507776t^3 + 170893152t^2 - 164410112t + 62195869.
\]

The revised sign list of the discriminant sequence of \( p_2(t) \) is

\[[1, 1, -1, -1].\tag{3.12}\]

The revised sign list of the discriminant sequence of \( p_3(t) \) is

\[[1, -1, -1, -1, 1, -1, 1, 1].\tag{3.13}\]

So the number of the sign changes of the revised sign list of (3.12) equals 1, then with Lemma 2.2 the equation \( p_2(t) = 0 \) has 2 distinct real roots. And by using the function "realroot()" [10, 11] in Maple 9.0, we can find that \( p_2(t) = 0 \) has 2 distinct real roots in the following intervals

\[[\frac{1}{2}, \frac{17}{32}], [\frac{77}{4}, \frac{617}{32}]\tag{3.14}\]

and no real root on interval \((\frac{11}{20}, \frac{7}{4})\).

And the number of the sign changes of the revised sign list of (3.13) equals 4, then from Lemma 2.3 the equation \( p_3(t) = 0 \) has 4 pairs distinct conjugate imaginary roots. It is to say that \( p_3(t) = 0 \) has no real root.

From (3.9), it is deduced easily that \( g(x_0) \) is the root of equation \( p_1(t) = 0 \). Namely, \( g(x_0) \) is the root of equation (1.6).

Further considering the proof above, we easily obtain the required result of the equality in (1.5).

Thus, the proof of Theorem 1.1 is completed. \( \square \)

**References**


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