SHARP INEQUALITIES FOR FACTORIAL n

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Abstract. Let n be a positive integer. We prove

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \le n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}},$$

with the best possible constants

$$\alpha = 1 - 2\pi e^{-2} = 0.149663... \ and \ \beta = 1/6 = 0.1666666....$$

This refines and extends a result of Sandor and Debnath, who proved that the double inequality holds with $\alpha = 0$ and $\beta = 1$.

1. INTRODUCTION

Stirling's approximation to n!,

$$n! \sim n^n e^{-n}\sqrt{2\pi n} = \alpha_n \tag{1.1}$$

plays a central role in statistical physics and probability theory. Inspired by this formula, many authors have made attempts to find a formula, which has an improvement over (1.1) and as simple as (1.1), to approximate n!. Such a typical result is due to Burnside[1]:

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} = \beta_n. \tag{1.2}$$

It is known that (1.2) has great superiority over (1.1). Formula (1.2) was rediscovered by Y. Weissman[9] and caused a lively debate in the American Journal of Physics in 1983, see[6]. Schuster found some other formulas to approximate n! but they are complicated and not easy to use[8]. In a recently paper Sandor and Debnath[7] found the following inequalities for $n \ge 2$:

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n}} \le n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1}}.$$

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This formula was rediscovered by Guo in a very newly paper[2]. In this short note we determine the largest number α and the smallest number β such that the inequalities

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \le n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}}$$

are valid for all positive integers n. Numerical computations indicate that the approximation

$$n! \sim \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} = \gamma_n \tag{1.3}$$

gives much more accurate values for $n!$ than α_n and β_n(see the table at the end of the paper). Throughout, we denote the gamma function Γ and its logarithmic derivative, known as psi or digamma function as

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du \ , \ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

for positive real numbers x , respectively.

In order to prove our main result we need to present two lemmas.

Lemma 1.1. *For $x \ge 1$ we have*

$$\sqrt{\pi}(x/e)^x(8x^3+4x^2+x+1/100)^{1/6} < \Gamma(x+1) < \sqrt{\pi}(x/e)^x(8x^3+4x^2+x+1/30).$$

This result is due to Karatsuba, see[4].

Lemma 1.2. *We have*

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2} - x\right) = -1/6.$$

Proof. Applying Stirling's formula, we get after a little simplification

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2} - x\right) = \lim_{x\to\infty}\frac{2\pi x^{2x}e^{-2x}}{(\Gamma(x+1))^2}\frac{2\pi x^{2x+2}e^{-2x} - x^3(\Gamma(x))^2}{2\pi x^{2x}e^{-2x}}$$

$$= \lim_{x\to\infty}\frac{2\pi - x^{1-2x}e^{2x}(\Gamma(x))^2}{2\pi(1/x)}.$$

By L'Hospital's rule this becomes

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)$$
$$=\frac{1}{\pi}\lim_{x\to\infty}x^{1-2x}e^{2x}(\Gamma(x+1))^2(\psi(x)-\log x+1/2x).$$

Using Stirling's formula again we get

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)=\lim_{x\to\infty}\frac{1-2x(\log x-\psi(x))}{(1/x)}.$$

By [5] we have

$$\log x-\psi(x)=\frac{1}{2x}+\frac{1}{12x^2}+\frac{\theta}{60x^4},$$

where $0<\theta<1$. Using this relation we find that

$$\lim_{x\to\infty}\left(\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x\right)=\lim_{x\to\infty}\frac{1-2\left(\frac{1}{2}+\frac{1}{12x}+\frac{\theta}{60x^4}\right)}{(1/x)}=-\frac{1}{6}.$$

□

2. MAIN RESULT

Our main result is the following theorem.

Theorem 2.1. *For any positive integer n the following double inequality holds*

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}}\le n!<\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}},$$

where the constants $\alpha=1-2\pi e^{-2}=0.149663...$ and $\beta=1/6=0.166666\ldots$ are best possible.

Proof. Set

$$h(x)=\frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2}-x,\quad x>0.$$

We show that h is strictly decreasing on $(0,\infty)$. Differentiating h, we get

$$h'(x)=\frac{4\pi(x/e)^{2x}(\log x-\psi(x))-(\Gamma(x))^2}{(\Gamma(x))^2}.$$

Hence in order to show that $h'(x) < 0$, it suffices to show

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}\right)^2 - 2x(\log x - \psi(x)) > 0.$$

From the left inequality of Lemma 1.1 we obtain for $x \geq 1$

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}\right)^2 > \left(8 + \frac{4}{x} + \frac{1}{x^2} + \frac{1}{100x^3}\right)^{1/3}.$$

In [3] it was proved that

$$x(\log x - \psi(x)) < \frac{1}{2} + \frac{1}{12x}.$$

Employing these two inequalities, we find that for $x \geq 1$

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}\right)^2 - 2x(\log x - \psi(x)) > \left(8 + \frac{4}{x} + \frac{1}{x^2} + \frac{1}{100x^3}\right)^{1/3} - 1 - \frac{1}{6x} > 0.$$

Hence, h is strictly decreasing on $(1, \infty)$. Using $h(1) = 2\pi e^{-2} - 1$ and $h(\infty) = -1/6$ by Lemma1.2, we get for any positive integer n

$$-1/6 = h(\infty) < h(n) = \frac{2\pi n^{2n+2} e^{-2n}}{(n!)^2} - n < h(1) = 2\pi e^{-2} - 1,$$

from which the proof follows. $\square$

The following table shows that γ_n has great superiority over α_n and β_n, where α_n, β_n and γ_n are as defined by (1.1), (1.2) and (1.3).

n	n!	α_n	β_n	γ_n
1	1	0.92213	1.02750	1.01015
2	2	1.91900	2.03331	2.00433
3	6	5.83620	6.07151	6.00541
4	24	23.50617	24.22261	24.01174
5	120	118.01916	120.91079	120.03673
6	720	710.07818	724.62384	720.15071
7	5040	4980.39583	5068.04888	5040.76647
8	40320	39902.39545	40517.97261	40324.65478
9	362880	359536.87284	364474.04470	362912.87998

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