SOME GAMMA FUNCTION INEQUALITIES

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ABSTRACT. In this paper we derive various new inequalities for the gamma function. Our results improve, refine and extend some previously known inequalities.

1. INTRODUCTION

The gamma Γ and psi ψ (or digamma)functions are defined as

$$\Gamma(x)=\int_0^{\infty} u^{x-1}e^{-u}du \ \ , \ \ \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$$

for positive real numbers x, respectively. For basic properties of these functions refer to [7, Chapter I]. We call the derivatives ψ', ψ'', ψ''', ... as polygamma functions. The gamma function play a key role in the proof of Stirling's formula

$$\lim_{n\to\infty}\frac{n!}{n^n e^{-n}\sqrt{n}}=\sqrt{2\pi},$$

see [10, Theorem 20], and has many applications in many fields of science such as special functions, number theory, statistics and physics. In particular, in the recent past many authors have studied these functions and they obtained many interesting inequalities for them, see([2], [3], [4], [5], [6], [8], [9], [12], [14], [15], [16], [17]). In this paper we aim at presenting several new inequalities for the gamma function. In what follows $c = 1.462632144968362\dots$ denotes the only positive zero of ψ-function(see[1, p.259,(6.3.19)].

In order to prove our main results we need to present some lemmas.

Lemma 1.1. *We have*

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\exp\left(\psi(x)e^{\psi(x)}-e^{\psi(x)}\right)}=\sqrt{2\pi}.$$

Proof. Using Stirling's formula we get

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\exp(\psi(x)e^{\psi(x)}-e^{\psi(x)})}=\sqrt{2\pi}\lim_{x\to\infty}\frac{x^x e^{-x}}{\sqrt{x}\exp(\psi(x)e^{\psi(x)}-e^{\psi(x)})}.$$

In [9] the author of this paper proved that

$$\lim_{x\to\infty}\left(x-e^{\psi(x)}\right)=1/2.$$

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Thus we have

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\exp(\psi(x)e^{\psi(x)}-e^{\psi(x)})}$$
$$=\sqrt{\frac{2\pi}{e}}\exp\left[\lim_{x\to\infty}\left(x\log x-\frac{1}{2}\log x-\psi(x)e^{\psi(x)}\right)\right].$$

In [5] it was proved that

$$\lim_{x\to\infty}[x(\log x-\psi(x))]=1/2.$$

Using this limit we obtain that

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\exp(\psi(x)e^{\psi(x)}-e^{\psi(x)})}$$
$$=\sqrt{2\pi}\exp\left[\lim_{x\to\infty}\left(\psi(x)(x-e^{\psi(x)})-\frac{1}{2}\log x\right)\right].$$

In [5, Eq.(2.2)], it is given

$$\log x-\frac{1}{x}<\psi(x)<\log x-\frac{1}{2x}\quad,\quad(x>0).$$

Using these bounds, we obtain

$$(x\log x-1)(1-e^{-1/(2x)})-\frac{1}{2}\log x<\psi(x)(x-e^{\psi(x)})-\frac{1}{2}\log x$$
$$<(x\log x-\frac{1}{2})(1-e^{-1/x})-\frac{1}{2}\log x.$$

It is not difficult to show that the limits of both of the bounds go to 0 when x tends to ∞, from which we get the desired limit. □

Lemma 1.2. *For $x>0$ we have*

$$\int_0^\infty\log\Gamma(t)dt=\frac{x}{2}-\frac{x^2}{2}+\frac{x}{2}\log\sqrt{2\pi}-(1-x)\log\Gamma(x)-\log G(x),$$

where G is Barnes G-function defined by the infinite product

$$G(z+1)=(2\pi)^{z/2}\exp\left(-\frac{1}{2}[(1+\gamma)z^2+z]\right)$$
$$\times\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k\exp\left(-z+\frac{z^2}{2k}\right)\right\}$$

and γ is Euler-Mascheroni constant given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577215664....$$

Barnes G-function satisfies the functional equation $G(1) = 1$ and $G(x+1) = \Gamma(x)G(x)$, see [11].

Let I be an interval, $s,t \in I$ and $f : I \to R$ continuous and strictly monotone. Then by mean value theorem for integration there exists a unique $\xi \in [s,t]$ for which

$$\frac{1}{t-s}\int_s^t f(u)du = f(\xi).$$

ξ is called *integral f-mean* of s and t, and denoted by

$$I_f = I_f(s,t) = f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)du\right).$$

I_f has the following property:

Lemma 1.3. *1. Let $f \in C^{(2)}(I)$, f is increasing, f' is decreasing and f''/f' is increasing. Then we have that $x \to I_f(x+s, x+t) - x$ is increasing.*

This is a known result and proved by Elezovic and Pecaric[13, Theorem 4].

2. MAIN RESULTS

In this section we collect our main results. Our first result gives new bounds in terms of psi function for the gamma function.

Theorem 2.1. *The following inequalities for the gamma function hold: For $x \geq c$*

$$\alpha \exp\left(\psi(x)e^{\psi(x)} - e^{\psi(x)}\right) < \Gamma(x) < \beta \exp\left(\psi(x)e^{\psi(x)} - e^{\psi(x)}\right),$$

where $\alpha = e\Gamma(c) = 2.4073190705\ldots$ and $\beta = \sqrt{2\pi} = 2.5066282746\ldots$ are best possible constants and for $0 < a \leq x \leq c$

$$e\Gamma(c)\exp[P(x)] \leq \Gamma(x) < \Gamma(a)\exp[P(x) - P(a)],$$

where $P(t) = e^{\psi(t)}(\psi(t) - 1)$.

Proof. Set

$$u(x) = \frac{\Gamma(x)}{\exp(\psi(x)e^{\psi(x)} - e^{\psi(x)})}.$$

Differentiation gives

$$u'(x) = \frac{\Gamma(x)\psi(x)(1 - \psi'(x)e^{\psi(x)})}{\exp\left(\psi(x)e^{\psi(x)} - e^{\psi(x)}\right)}.$$

Since $\psi'(x)e^{\psi(x)} < 1$ by [9, Lemma 1.2] for $x > 0$, $\psi(x) \geq 0$ for $x \geq c$ and $\psi(x) \leq 0$

for $x \leq c$, u is strictly increasing on $[c, \infty)$ and strictly decreasing on $(0, c]$. Since $\lim_{x\to\infty} u(x) = \sqrt{2\pi}$ by Lemma 1.1, we have for $x \geq c$

$$e\Gamma(c) = u(c) \leq u(x) < \lim_{x\to\infty} u(x) = \sqrt{2\pi}, \quad x \geq c.$$

This proves the first part of Theorem 2.1. Similarly if $0 < a \leq x \leq c$ we have $u(c) < u(x) \leq u(a)$.Using the definition of u, we prove the second part of Theorem 2.1. □

The author of present paper[9] proved the inequalities for the gamma function:

$$\Gamma(x + 1/2) > \Gamma(c)x^x e^{-x+1},\ x \geq 1/2$$

and

$$\Gamma(x + c - 1) \leq \Gamma(c)x^{kx} e^{-kx}, x \geq c - 1$$

where $k = 6e^\gamma/\pi^2 = 1.08...$ and γ is Euler constant. The following theorem extend, refine and improve these inequalities.

Theorem 2.2. *For $x > 1$ the following inequalities hold*

$$\Gamma(x + c - 1) < \Gamma(c)\left(x^x e^{1-x}\right)^{\varphi(1)} \tag{2.1}$$

and

$$\Gamma(x + 1/2) > \Gamma(c)\left(x^x e^{1-x}\right)^{\varphi(x)}, \tag{2.2}$$

and for $0 < a \leq x \leq 1$

$$\Gamma(x + c - 1) < \Gamma\left(\psi^{-1}(\log x)\right)\left(a^{-a}x^x e^{a-x}\right)^{\varphi(x)} \tag{2.3}$$

and

$$\Gamma(x + a + \psi^{-1}(\log a) - 1) \geq \Gamma\left(\psi^{-1}(\log a)\right)\left(a^{-a}x^x e^{a-x}\right)^{\varphi(a)}, \tag{2.4}$$

where

$$\varphi(x) = \frac{1}{x\psi'\left(\psi^{-1}(\log x)\right)}. \tag{2.5}$$

Proof. Applying mean value theorem to $\log(\Gamma(x))$ on the interval $(u, u + 1)$, there exists a θ_0 such that $0 < \theta_0 = \theta_0(u) < 1$ and

$$\psi(u + \theta_0(u)) = \log u. \tag{2.6}$$

It is easy to deduce from this identity that

$$\theta_0\left(e^{\psi(u)}\right) = u - e^{\psi(u)} \quad . \tag{2.7}$$

Integrating both sides of (2.6) on the interval $1 \le u \le x$ we get

$$\int_1^x \psi(u+\theta_0(u))du = x\log x - x + 1. \tag{2.8}$$

Inducing the change of variable $u = e^{\psi(t)}$ here and using (2.6), we obtain

$$\int_c^{x+\theta_0(x)} \psi(t)\psi'(t)e^{\psi(t)}dt = x\log x - x + 1, \quad x > 1. \tag{2.9}$$

In [9, Lemma 1.1], it was proved that $(\psi'(x))^2 + \psi''(x) > 0$ for $x > 0$, so that mapping $t \to \psi'(t)e^{\psi(t)}$ is strictly increasing on $(0,\infty)$. Since $\psi(t) \ge 0$ for $t \ge c$ we conclude from (2.9) that

$$x\psi'(x+\theta_0(x))\int_c^{x+\theta_0(x)} \psi(t)dt > x\log x - x + 1, \;\; x > 1$$

or

$$x\log x - x + 1 \le x\psi'\left(\psi^{-1}(\log x)\right)\left[\log\left(\Gamma(x+\theta_0(x)) - \log\left(\Gamma(c)\right)\right]\quad , \; x > 1\right.$$

Simplifying this we get for $x > 1$

$$\Gamma(x+\theta_0(x)) > \Gamma(c)\left(x^x e^{1-x}\right)^{\varphi(x)}. \tag{2.10}$$

In [9] it is proved that $x \to \theta_0(x)$ is strictly increasing on $(0,\infty)$ and $\lim_{x\to\infty}\theta_0(x) = 1/2$. Hence we obtain for $x > 1$

$$\Gamma(c)\left(x^x e^{1-x}\right)^{\varphi(x)} \le \Gamma(x+1/2),$$

proving (2.2). From (2.9) we write

$$x\log x - x + 1 \ge \psi'(c)\left[\log\left(\Gamma(x+\theta_0(x)) - \log\left(\Gamma(c)\right)\right]\quad , \; x > 1\right.$$

or

$$\Gamma(c)\left(x^x e^{1-x}\right)^{\varphi(1)} \ge \Gamma(x+\theta_0(x)), \;\; x > 1. \tag{2.11}$$

Since $x + \theta_0(x) > x + \theta_0(1) = x + c - 1$ for $x > 1$, we conclude for $x > 1$ that

$$\Gamma(c)\left(x^x e^{1-x}\right)^{\varphi(1)} \ge \Gamma(x+c-1).$$

This proves (2.1).

Now we let $0 < a \le x \le 1$. Integrating both sides of (2.6) on the interval $a \le u \le x \quad (x \le 1)$, we find that

$$\int_a^x \psi(u+\theta_0(u))du = x\log x - x - a\log a + a.$$

Making the change of variable $u = e^{\psi(t)}$ here, we get

$$\int_{a+\theta_0(a)}^{x+\theta_0(x)} (-\psi(t))\psi'(t)e^{\psi(t)}dt = -x\log x + x + a\log a - a. \tag{2.12}$$

Since $-\psi(t) \geq 0$ for t$\leq c$, we get from (2.12) that

$$-x\log x + a\log a - a \leq x\psi'\left(\psi^{-1}(\log x)\right)\int_{a+\theta_0(a)}^{x+\theta_0(x)} (-\psi(t))dt.$$

Simplifying this identity we get for $0 < a \leq x \leq 1$

$$\Gamma\left(\psi^{-1}(\log a)\right)\left(x^x e^{a-x}a^{-a}\right)^{\varphi(x)} \geq \Gamma(x+\theta_0(x)). \tag{2.13}$$

Since $x + \theta_0(x) \geq x + \theta_0(1) = x + c - 1$ this proves (2.3). Similarly, from (2.12) we get

$$-x\log x + a\log a - a \geq a\psi'\left(\psi^{-1}(\log a)\right)\int_{a+\theta_0(a)}^{x+\theta_0(x)} (-\psi(t))dt$$

or

$$\Gamma(x+\theta_0(x)) \geq \Gamma\left(\psi^{-1}(\log a)\right)\left(x^x e^{a-x}a^{-a}\right)^{\varphi(a)}. \tag{2.14}$$

But since $a \leq x \leq 1$ and θ_0 is strictly increasing this gives

$$\Gamma(x + \psi^{-1}(\log a) - a) \geq \Gamma\left(\psi^{-1}(\log a)\right)\left(a^{-a}x^x e^{a-x}\right)^{\varphi(a)}.$$

This completes the proof of Theorem 2.2. □

In 2004, in [9, Theorem 2.1] the author of this paper proved the following inequalities for $x \geq c$: If $\alpha = 1$ and $\beta = 6e^{\gamma}/\pi^2$, then the double inequality

$$\Gamma(c)\exp\left(\alpha\left(e^{\psi(x)} - \psi(x)e^{\psi(x)} + 1\right)\right) < \Gamma(x) < \Gamma(c)\exp\left(\beta\left(e^{\psi(x)} - \psi(x)e^{\psi(x)} + 1\right)\right) \tag{2.15}$$

holds. Alzer and Grinshpan [2, Theorem 4.3] refined the right hand side of this inequality. More precisely, they proved that the left hand side of (2.15) is sharp(in the sense that α can not be replaced by a larger number than 1) but number β in the right hand side can be replaced by a smaller number. Namely, they proved that (2.15) holds even for $\beta = 1/\psi'(c) = 1.0334....$ In the following theorem we improve the left hand side of (2.15).

Theorem 2.3. *We have for* $x \geq c$

$$\Gamma(c)\exp\left(\phi(x)\left(e^{\psi(x)} - \psi(x)e^{\psi(x)} + 1\right)\right) < \Gamma(x)$$
$$< \Gamma(c)\exp\left(\phi(c)\left(e^{\psi(x)} - \psi(x)e^{\psi(x)} + 1\right)\right) \quad , \tag{2.16}$$

and for $0 < a \leq x \leq c$

$$\Gamma(a)\exp\left(\phi(a)\left(\psi(x)e^{\psi(x)} - e^{\psi(x)} - \psi(a)e^{\psi(a)} + e^{\psi(a)}\right)\right) \leq \Gamma(x)$$
$$\leq \Gamma(a)\exp\left(\phi(x)\left(\psi(x)e^{\psi(x)} - e^{\psi(x)} - \psi(a)e^{\psi(a)} + e^{\psi(a)}\right)\right) \tag{2.17}$$

where

$$\phi(x) = \frac{1}{\psi'(x)e^{\psi(x)}}.$$

Proof. Replacing x by $e^{\psi(x)}$ in (2.10), we get for $x \geq c$ the left of (2.16). Similarly replacing x by $e^{\psi(x)}$ in (2.11) proves the right of (2.16). Substituting $e^{\psi(x)}$ and $e^{\psi(a)}$ for x and a in (2.13) and (2.14), respectively, we obtain the right and the left of (2.17). □

We note that since $\phi(x) > 1$ by [9, Lemma 1.2], the left-hand side of (2.16) improves the left of (2.15).

The second Gautschi-Kershaw inequality states that

$$\exp[(1-s)\psi(x+\sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp[(1-s)\psi(x+(s+1)/2)]$$

where $0 < s < 1$ and $x > 0$, see([16], [12]). In the following theorem we give an elementary proof of an extended form of these inequalities.

Theorem 2.4. *Let* x *and* y *be positive real numbers. Then we have*

$$\exp\left((x-y)\psi\left(\frac{x-y}{\log(x+1)-\log(y+1)} - 1\right)\right) < \frac{\Gamma(x)}{\Gamma(y)}$$
$$< \exp\left((x-y)\psi\left(\frac{x+y}{2}\right)\right). \tag{2.18}$$

Proof. The psi- function have the following series representation:

$$\psi(u) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+u}\right)(u > 0), \tag{2.19}$$

where γ is Euler constant. Integrating both sides of (2.19) over $1 \leq u \leq x$, we get

$$\log(\Gamma(x)) = -\gamma(x-1) + \sum_{k=1}^{\infty}\left(\frac{x-1}{k} - \log(x+k) + \log(k+1)\right). \tag{2.20}$$

From (2.20) we find that

$$\log(\Gamma(x)) - \log(\Gamma(y)) = -\gamma(x-y) + \sum_{k=1}^{\infty}\left(\frac{x-y}{k} - (f(k+x) - f(k+y))\right), \tag{2.21}$$

where $f(t) = \log t$. By mean value theorem we can write

$$f(k+x) - f(k+y) = (x-y)f'(k+\theta(k)) \tag{2.22}$$

with $\theta(k)$ is a number between x and y. Using (2.22) we write (2.21) as

$$-\frac{\log(\Gamma(x)) - \log(\Gamma(y))}{x-y} = \gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k+\theta(k)} - \frac{1}{k}\right). \tag{2.23}$$

We find from (2.22) that

$$\theta(k) = h^{-1}\left(\frac{1}{x-y}\int_{k+y}^{k+x} h(u)du\right) - k,$$

where $h(t) = -1/t$. Since $h'(t) > 0$, $h''(t) < 0$ *and* $(h'''(t)/h''(t))' > 0$, mapping $k \to \theta(k)$ is strictly increasing on $(0, \infty)$ by Lemma 1.3. Hence, we conclude from (2.23) that

$$\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k+\theta(\infty)} - \frac{1}{k}\right) < -\frac{\log(\Gamma(x)) - \log(\Gamma(y))}{x-y} < \gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k+\theta(1)} - \frac{1}{k}\right).$$

After some manipulation this can be written by (2.19) as

$$\psi(\theta(1)) < \frac{\log(\Gamma(x)) - \log(\Gamma(y))}{x-y} < \psi(\theta(\infty)).$$

From (2.22) we get

$$\theta(k) = \frac{x-y}{\log(x+k) - \log(y+k)} - k. \tag{2.24}$$

It is not difficult to show that

$$\lim_{k\to\infty} \theta(k) = \frac{x+y}{2}.$$

From (2.24) we have

$$\theta(1) = \frac{x-y}{\log(x+1) - \log(y+1)} - 1.$$

Therefore we conclude that

$$\psi\left(\frac{x-y}{\log(x+1)-\log(y+1)}-1\right) < \frac{\log\left(\Gamma(x)\right)-\log\left(\Gamma(y)\right)}{x-y} < \psi\left(\frac{x+y}{2}\right),$$

proving Theorem 2.4. □

Theorem 2.5. *For $x \geq c$ the following inequalities hold:*

$$\Gamma(x+\alpha) \leq x^x e^{-x}\sqrt{2\pi} \leq \Gamma(x+\beta),$$

where $\alpha = 0.5$ and $\beta = \Gamma^{-1}(c^c e^{-c}\sqrt{2\pi}) - c$ =0.566334355667106. . . . are best possible constants.

Proof. We define for $u \geq 0$

$$F(u) = \int_c^u \log(\Gamma(t))dt.$$

Applying mean value theorem to F on $[x, x+1]$, we get an $\varepsilon = \varepsilon(x)$ such that $0 < \varepsilon(x) < 1$ and

$$F(x+1) - F(x) = \log[\Gamma(x+\varepsilon(x))].$$

Using Lemma 1.2 and the functional equations $G(1) = 1$ and $G(x+1) = \Gamma(x)G(x)$ this can be written as

$$\Gamma(x+\varepsilon(x)) = x^x e^{-x}\sqrt{2\pi}. \tag{2.25}$$

Now we shall prove that ε has the following properties:
a) ε is strictly decreasing on $x \geq \psi^{-1}(\log c)$=1.934356051030967. . . .
b) $\lim_{x\to\infty} \varepsilon(x) = 1/2$.
In what follows we denote k = $\psi^{-1}(\log c)$=1.934356051030967. Differentiation of (2.25) yields

$$\varepsilon'(x) = \frac{\log x}{\psi\left(\Gamma^{-1}(x^x e^{-x}\sqrt{2\pi})\right)}.$$

Since the mapping $x \to e^{\psi(x)}$ is a bijection on $(0, \infty)$, in order to show $\varepsilon'(x) < 0$ for $x \geq c$ it is enough to show for $x \geq k$

$$\varepsilon'(e^{\psi(x)}) = \frac{\psi(x)}{\psi\left(\Gamma^{-1}(g(x))\right)} - 1 < 0,$$

where

$$g(x) = \sqrt{2\pi}\exp\left(\psi(x)e^{\psi(x)} - e^{\psi(x)}\right)$$

or equivalently

$$\frac{\Gamma(x)}{\sqrt{2\pi}\exp[\psi(x)e^{\psi(x)} - \psi(x)]} < 1,$$

but this is obviously true by Theorem 2.1. Hence, ε is strictly decreasing on $[c, \infty)$. Putting $x+1$ for x in (2.25) and using the functional equation $\Gamma(x+1) = x\Gamma(x)$ for the gamma function we get

$$(x+\varepsilon(x+1))\Gamma(x+\varepsilon(x+1)) = \sqrt{2\pi}(x+1)^{x+1}e^{-(x+1)}$$

or

$$\varepsilon(x+1) = \frac{\sqrt{2\pi}(x+1)^{x+1}e^{-(x+1)}}{\Gamma(x+\varepsilon(x+1))} - x.$$

Since ε is strictly decreasing and bounded it has a limit as x tends to infinity. So by the help of (2.25) we obtain

$$\lim_{x\to\infty} \varepsilon(x) = \lim_{x\to\infty} \left[\frac{\sqrt{2\pi}(x+1)^{x+1}e^{-(x+1)}}{\Gamma(x+\varepsilon(x))}) - x \right]$$

$$= \lim_{x\to\infty} \left[e^{-1}\frac{(1+x)^{x+1}}{x^x} - x \right].$$

It is easy to show that this limit has value $1/2$. Thus we get for $x \geq c$

$$0.5 = \varepsilon(\infty) < \varepsilon(x) \leq \varepsilon(c) = \Gamma^{-1}\left(c^c e^{-c}\sqrt{2\pi}\right) - c = 0.56633435566\ldots.$$

Now the proof is completed by using (2.25). □

References

[1] Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Functions*, *Dover, New York*, 1965.

[2] Alzer, H., and Grinshpan,A. Z., Inequalities for the gamma and q-gamma functions, *J. Approx. Theory*, (in press).

[3] Alzer, H., and Berg, C., Some Classes of completeley monotonic functions II, *The Ramanujan Journal*, 11(2006), no.2, 225-248.

[4] Alzer, H., On Ramanujan's Double inequality for the gamma function, *Bull. London Math. Soc.* 35(2003), 601-607.

[5] Alzer, H., On some inequalities for the gamma and psi functions, *Math. Comp.* 66(1997), no.217, 373-389.

[6] Anderson,G. D., Barnard, R. W., Richards, K. C., Vamanamurthy, M. K., and Vuorinen,M., Inequalities for zero-balanced hypergeometric functions, *Trans Amer. Math. Soc.,* 347(1995), 1713-1723.

[7] Andrews, G., Askey, R., and Toy, R., Special functions , Encyclopedia of Mathematics and its Applications, v.71, Cambridge U. Pres, 1999.

[8] Batir, N., An interesting double inequality for Euler's gamma function, *JIPAM. J. Inequal. Pure Appl.* , 5(2004), no. 4, Article 97,3 pp.

[9] Batir, N., Some new inequalities for gamma and polygamma functions, *JIPAM. J. Inequal. Pure Appl.* , 6(2005), no.4, Article 103, 9 pp.

[10] Buck, R. C., Advanced Calculus, McGraw-Hill Kogakusha, Third Ed., 1978.

[11] Choi, I.J., Srivastava, H. M., and Adamchik, V. S., Multiple Gamma and related functions, *Appl. Math. Comp.* V.134 (2003),515-533.

[12] Elezovic, N., Giordano, C.and Pecaric, J., The best bounds in Gautschi's inequality. *Math. Inequal. Appl.* 3 (2000), 239-252.

[13] Elezovic, N., and Pecaric, J., Differential and Integral f-means and applications to digamma functions, *Math. Inequal. Appl.* 3(2000), no.2, 189-196.

[14] Grinshpan, A. Z., and Ismail, M. E. H., Completely monotonic functions involving the gamma and q-gamma functions, *Proc. Amer. Math. Soc.*, 134(2006), no.4, 1153-1160.

[15] Karatsuba, E. A., On the Asymptotic representation of the Euler gamma function by Ramanujan, *J. Comp. Appl. Math.* 135.2(2001), 225-240.

[16] Kershaw, D., Some extensions of W. Gautschi's inequalities for the gamma function, *Math. Comp.*, 41(1983), 607- 611.

[17] Qui, S. L., and Vuorinen,M., Some properties of the gamma and psi functions, with applicationa, *Math. Comp.* 74 (2005), no. 250, 723-742.

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