JORDAN'S INEQUALITY: REFINEMENTS, GENERALIZATIONS, APPLICATIONS AND RELATED PROBLEMS

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ABSTRACT. This is an expository article. Some developments on refinements, generalizations, applications of Jordan's inequality and related problems, including some estimates for three classes of complete elliptic integrals and several proofs of Wilker's inequality, are summarized.

1. REFINEMENTS OF JORDAN'S INEQUALITY

1.1. **Jordan's inequality.** The well-known Jordan's inequality (see [2, 9], [5, p. 143], [23, p. 269] and [27, p. 33]) reads that

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \tag{1.1}$$

for $0 < |x| \le \frac{\pi}{2}$. The equality in (1.1) is valid if and only if $x = \frac{\pi}{2}$.

Note that the origin of Jordan's inequality is not found in the references listed in this paper. So, it is unknown that why inequality (1.1) is due to Jordan and to which Jordan.

1.2. **Kober's inequality.** In [23, pp. 274–275], an inequality due to Kober [20, p. 22] was given:

$$1 - \frac{2}{\pi}x \le \cos x \le 1 - \frac{x^2}{\pi}, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{1.2}$$

In [21] and [22, p. 313], it was given that for $x \in [0, \pi]$,

$$\cos x \le 1 - \frac{2}{\pi^2}x^2. \tag{1.3}$$

The left hand side inequalities in (1.1) and (1.2) are equivalent, since they can be deduced from each other via the transformation $x \to \frac{\pi}{2} - x$.

1.3. **Redheffer's inequality.** In [44, 45], it was proposed that

$$\frac{\sin x}{x} \ge \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \ne 0. \tag{1.4}$$

In [49], inequality (1.4) was proved as follows. For $x \ge 1$,

$$\begin{aligned} \frac{1 - x^2}{1 + x^2} - \frac{\sin(\pi x)}{\pi x} &= \frac{1 - x^2}{1 + x^2} + \frac{\sin[\pi(x - 1)]}{\pi(x - 1)} \cdot \frac{x - 1}{x} \\ &\le \frac{1 - x^2}{1 + x^2} + \frac{x - 1}{x} = -\frac{(1 - x)^2}{x(1 + x^2)} \le 0. \end{aligned}$$

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For $0<x<1$, since $\frac{\sin(\pi x)}{\pi x}=\prod_{k=1}^{\infty}\bigl(1-\frac{x^2}{k^2}\bigr)$, it is enough to prove that $(1+x^2)P_n\ge 1$ for $n\ge2$, where $P_n=\prod_{k=2}^{n}\bigl(1-\frac{x^2}{k^2}\bigr)$. Actually, by a simple induction argument based on the relation $P_{n+1}=\bigl[1-\frac{x^2}{(n+1)^2}\bigr]P_n$, it is deduced that $(1+x^2)P_n\ge1+\frac{x^2}{n}$ for $0<x<1$.

1.4. **Caccia's inequality.** In [26], it was proposed that

$$\sin\theta\ge\frac{2}{\pi}\theta+\frac{1}{12\pi}\theta\bigl(\pi^2-4\theta^2\bigr) \tag{1.5}$$

for $\theta\in\bigl[0,\frac{\pi}{2}\bigr]$. In [1], by finding the minimum of the function

$$\begin{cases}1, & x=0,\\ x^{-1}\sin x+\frac{x^2}{3\pi}, & x\in\bigl(0,\frac{\pi}{2}\bigr],\end{cases}$$

inequality (1.5) was proved by U. Abel. Meanwhile, inequality (1.5) is improved in [1] by D. Caccia as

$$\sin\theta\ge\frac{2}{\pi}\theta+\frac{1}{\pi^3}\theta\bigl(\pi^2-4\theta^2\bigr) \tag{1.6}$$

for $\theta\in\bigl[0,\frac{\pi}{2}\bigr]$. Inequality (1.6) is slightly stronger than (1.5) and is sharp in the sense that $\frac{1}{\pi^3}$ cannot be replaced by a larger constant.

1.5. **Prestin's inequality.** In [30] and [23, p. 270], the following inequality is given: For $0<|x|\le\frac{\pi}{2}$,

$$\left|\frac{1}{\sin x}-\frac{1}{x}\right|\le1-\frac{2}{\pi}. \tag{1.7}$$

1.6. **Refinements of Jordan's and Kober's inequality by Taylor's formula.** In [19, pp. 101–102], [22, p. 313] and [23, p. 269], the following inequalities are mentioned: For $x\in\bigl[0,\frac{\pi}{2}\bigr]$,

$$x-\frac{1}{6}x^3\le\sin x\le x-\frac{1}{6}x^3+\frac{1}{120}x^5, \tag{1.8}$$

$$1-\frac{1}{2}x^2\le\cos x\le1-\frac{1}{2}x^2+\frac{1}{24}x^4, \tag{1.9}$$

$$(-1)^n\left[\sin x-\sum_{k=1}^{n}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}\right]\le\frac{x^{2n+1}}{(2n+1)!}, \tag{1.10}$$

$$(-1)^{n+1}\left[\cos x-\sum_{k=0}^{n}(-1)^{k}\frac{x^{2k}}{(2k)!}\right]\le\frac{x^{2n+2}}{(2n+2)!}. \tag{1.11}$$

In [25], inequality (1.8) was applied to obtain the lower and upper estimations of $\zeta(3)$ by $\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3}=\frac{1}{4}\int_0^{\pi/2}\frac{x(\pi-x)}{\sin x}\,\mathrm{d}x=\frac{7}{8}\zeta(3)$.

1.7. **Refinements of Jordan's inequality by a method of auxiliary functions.** In [34], with the help of the following two auxiliary functions $\cos x-1+\frac{2}{\pi}x-\alpha x(\pi^2-x^2)$ and $\cos x-1+\frac{2}{\pi}x-\beta x(\pi-2x)$ for $x\in\bigl[0,\frac{\pi}{2}\bigr]$ with undetermined positive constants α and β, Kober's inequality (1.2) was refined: For $x\in\bigl[0,\frac{\pi}{2}\bigr]$,

$$1-\frac{2}{\pi}x+\frac{\pi-2}{\pi^2}x(\pi-2x)\le\cos x\le1-\frac{2}{\pi}x+\frac{2}{\pi^2}x(\pi-2x), \tag{1.12}$$

$$1-\frac{2}{\pi}x+\frac{\pi-2}{2\pi^3}x\bigl(\pi^2-4x^2\bigr)\le\cos x\le1-\frac{2}{\pi}x+\frac{2}{\pi^3}x\bigl(\pi^2-4x^2\bigr). \tag{1.13}$$

These two double inequalities are sharp in the sense that the constants $\frac{\pi-2}{\pi^2}$, $\frac{2}{\pi^2}$, $\frac{\pi-2}{2\pi^3}$ and $\frac{2}{\pi^3}$ connot be replaced by larger or smaller ones respectively.

Inequality (1.12) is better than (1.13). Inequality (1.12) may be rewritten as

$$1-\frac{4-\pi}{\pi}x-\frac{2(\pi-2)}{\pi^2}x^2\le\cos x\le 1-\frac{4}{\pi^2}x^2. \tag{1.14}$$

Inequality (1.14) is stronger than (1.3) on $\left[0,\frac{\pi}{2}\right]$. Replacing x by $\frac{\pi}{2}-x$ in (1.14) gives

$$x-\frac{2(\pi-2)}{\pi^2}x^2\le\sin x\le\frac{4}{\pi}x-\frac{4}{\pi^2}x^2,\quad x\in\left[0,\frac{\pi}{2}\right]. \tag{1.15}$$

In [37], by considering auxiliary functions $\sin x-\frac{2}{\pi}x-\alpha x\big(\pi^2-4x^2\big)$, $\sin x-\frac{2}{\pi}x-\beta x^2(\pi-2x)$ and $\sin x-\frac{2}{\pi}x-\theta x(\pi-2x)$ on $\left[0,\frac{\pi}{2}\right]$, inequality (1.6) and the following inequalities are obtained:

$$\sin x\le\frac{2}{\pi}x+\frac{\pi-2}{\pi^3}x\big(\pi^2-4x^2\big), \tag{1.16}$$

$$\sin x\ge\frac{2}{\pi}x+\frac{4}{\pi^3}x^2(\pi-2x), \tag{1.17}$$

$$\frac{2}{\pi}x+\frac{\pi-2}{\pi^2}x(\pi-2x)\le\sin x\le\frac{2}{\pi}x+\frac{2}{\pi^2}x(\pi-2x), \tag{1.18}$$

where the constants $\frac{\pi-2}{\pi^3}$, $\frac{4}{\pi^3}$, $\frac{\pi-2}{\pi^2}$ and $\frac{2}{\pi^2}$ are the best possible. Inequality (1.18) can be rewritten as (1.15). Combination of (1.6) and (1.16) leads to

$$\frac{3}{\pi}x-\frac{4}{\pi^3}x^3\le\sin x\le x-\frac{4(\pi-2)}{\pi^3}x^3,\quad x\in\left[0,\frac{\pi}{2}\right]. \tag{1.19}$$

Inequality (1.15) and (1.19) are not included on $\left[0,\frac{\pi}{2}\right]$ each other. Inequality (1.17) is weaker than the left hand side inequality in (1.19) and can not compare with the left hand side inequality of (1.15).

In [32], by constructing suitable auxiliary functions, inequality (1.16) or the right hand side inequality of (1.19), the double inequality (1.18) or (1.15), inequality (1.17), the double inequality (1.12) or (1.14), the double inequality (1.13) and their sharpness are verified again. Employing these inequalities, it is deduced that

$$\frac{4}{3}<\int_0^{\pi/2}\frac{\sin x}{x}\,\mathrm{d}x<\frac{\pi+1}{3}\quad\text{and}\quad\frac{1}{2}<\int_0^{\pi/2}\frac{1-\cos x}{x}\,\mathrm{d}x<\frac{6-\pi}{4}. \tag{1.20}$$

In [39], Jordan's inequality was interpreted geometrically, inequalities (1.6) and (1.16) or their variant (1.19) and inequality (1.12) or (1.14) were proved once more by considering suitable auxiliary functions. From (1.19) and the symmetry and period of $\sin x$, it is deduced that

$$\begin{aligned}\frac{4}{\pi^3}x^3-\frac{12}{\pi^2}x^2+\frac{9}{\pi}x-1&\le\sin x\\&\le\frac{4(\pi-2)}{\pi^3}x^3-\frac{12(\pi-2)}{\pi^2}x^2+\frac{11\pi-24}{\pi}x+8-3\pi\end{aligned} \tag{1.21}$$

on $\left[\frac{\pi}{2},\pi\right]$ and

$$\frac{7}{6}-\ln 2<\int_{\pi/2}^{\pi}\frac{\sin x}{x}dx<\frac{13\pi-32}{6}+(8-3\pi)\ln 2. \tag{1.22}$$

1.8. **Refinements of Jordan's inequality by L'Hôspital's rule.** In [3, Theorem 1.25], the following monotonic form of L'Hôspital's rule was put forwarded.

Lemma 1. *Let f and g be continuous on $[a,b]$ and differentiable in (a,b) such that $g'(x)\ne 0$ in (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) in (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing) in (a,b).*

In [57], by using Lemma 1, inequalities (1.6), (1.12), (1.13), (1.16) and (1.18) were recovered once more.

1.9. **Some other results.**

1.9.1. In [33], the double inequality (1.19) was verified once again. Moreover, among other things, several inequalities and integrals related to $\frac{\sin x}{x}$ are constructed by using the well known Tchebysheff's integral inequality, for example,

$$\left(\frac{\sin t}{t}\right)^2+2\left(\frac{\sin t}{t}\right)\ge 4\left(\frac{1-\cos t}{t^2}\right)+\cos t,\quad t\in[0,\pi] \tag{1.23}$$

and

$$\int_0^t\left(\frac{x}{\sin x}\right)^2\,\mathrm{d}\,x<2\tan\left(\frac{t}{2}\right)+\frac{2}{3}\tan^3\left(\frac{t}{2}\right),\quad t\in\left(0,\frac{\pi}{2}\right]. \tag{1.24}$$

1.9.2. In [50, 51], by considering the logarithmic concavity of $\frac{\sin x}{x}$ and the logarithmic convexity of $\frac{\tan x}{x}$ and by using Jensen's inequality, it was obtained that

$$\begin{aligned}\left|\prod_{i=1}^n\tan x_i\right|&\ge\left|\prod_{i=1}^n x_i\left[\frac{\tan\frac{\sum_{i=1}^n|x_i|}{n}}{\frac{\sum_{i=1}^n|x_i|}{n}}\right]^n\right|\\&>\left|\prod_{i=1}^n x_i\right|>\left|\prod_{i=1}^n x_i\left[\frac{\sin\frac{\sum_{i=1}^n|x_i|}{n}}{\frac{\sum_{i=1}^n|x_i|}{n}}\right]^n\right|\ge\left|\prod_{i=1}^n\sin x_i\right|\end{aligned} \tag{1.25}$$

holds for $0<|x_i|<\frac{\pi}{2}$, $1\le i\le n$ and $n\in\mathbb{N}$. For $0<\beta<\alpha$ and $0<|\alpha x|<\frac{\pi}{2}$,

$$\frac{2}{\pi}\le\left|\frac{\sin(\beta x)}{\alpha x\sin\frac{\beta\pi}{2\alpha}}\right|\le\left|\frac{\sin(\alpha x)}{\alpha x}\right|<\left|\frac{\sin(\beta x)}{\beta x}\right|<1, \tag{1.26}$$

$$\frac{|\tan(\alpha x)|}{\alpha|x|}>\frac{|\tan(\beta x)|}{\beta|x|}>1>\frac{|\sin(\beta x)|}{\beta|x|}>\frac{|\sin(\alpha x)|}{\alpha|x|}>\frac{|\sin(\beta x)|}{\alpha|x|}\csc\frac{\beta\pi}{2\alpha}. \tag{1.27}$$

1.9.3. Let

$$p(\theta)=\begin{cases}\left(\frac{\pi^2}{8}-\frac{1}{2}\theta\right)\sec^2\theta-\theta\tan\theta-\frac{1}{2}, & \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\\ 0, & \theta=\pm\frac{\pi}{2},\end{cases} \tag{1.28}$$

$$q(\theta)=\begin{cases}\frac{2}{\cos^2\theta}\int_\theta^{\pi/2}t\cos^2t\,\mathrm{d}\,t, & \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\\ 0, & \theta=\pm\frac{\pi}{2},\end{cases} \tag{1.29}$$

$$\phi(\theta)=\begin{cases}\frac{\pi}{4}\big(\theta\sec^2\theta+\tan\theta\big)-2\tan\theta\sec\theta, & \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\\ \pm1, & \theta=\pm\frac{\pi}{2}.\end{cases} \tag{1.30}$$

These three functions originate from estimates of eigenvalues of Laplace operator on compact Riemannian manifolds. Their monotonicity and estimates have been investigated. For more detailed information, please refer to [16, 36, 41] and the references therein.

1.9.4. Some results in [4, 7] may be interesting.

2. Refinements of Jordan's inequality and L. Yang's inequality

2.1. **L. Yang's inequality.** In [55, pp. 116–118], an inequality due to L. Yang states that inequality

$$\cos^2(\lambda A)+\cos^2(\lambda B)-2\cos(\lambda A)\cos(\lambda B)\cos(\lambda\pi)\ge\sin^2(\lambda\pi) \tag{2.1}$$

is valid for $0\le\lambda\le1$, $A>0$ and $B>0$ with $A+B\le\pi$, where the equality holds if and only if $\lambda=0$ or $A+B=\pi$.

Inequality (2.1) has been generalized in [59, 60] and the references therein.

2.2. **Debnath-Zhao's result.** In [8], inequalities (1.5) and (1.6) or the left hand side inequality in (1.19) were recovered. However, it seems that the authors of [8] did not compare explicitly their recovered results (1.5) and (1.6).

As an application of (1.6), with the help of

$$\begin{aligned}\sin^2(\lambda\pi)\le\cos^2(\lambda A_i)+\cos^2(\lambda A_j)&\\ -2\cos(\lambda A_i)\cos(\lambda A_j)\cos(\lambda\pi)&\triangleq H_{ij}\le4\sin^2\left(\frac{\lambda}{2}\pi\right)\end{aligned} \tag{2.2}$$

in [59] and [60, (2.13)], where $0\le\lambda\le1$ and $A_i>0$ with $\sum_{i=1}^nA_i\le\pi$ for $n\ge2$, L. Yang's inequality (2.1) was generalized to

$$\binom{n}{2}\lambda^2\big(3-\lambda^2\big)^2\cos^2\left(\frac{\lambda}{2}\pi\right)\le\sum_{1\le i<j\le n}H_{ij}\le\binom{n}{2}\lambda^2\pi^2. \tag{2.3}$$

2.3. **Özban's result.** In [28], the author gave a new refined form of Jordan's inequality for $0<x\le\frac{\pi}{2}$

$$\frac{\sin x}{x}\ge\frac{2}{\pi}+\frac{1}{\pi^3}\big(\pi^2-4x^2\big)+\frac{4(\pi-3)}{\pi^3}\left(x-\frac{\pi}{2}\right)^2 \tag{2.4}$$

with equality if and only if $x=\frac{\pi}{2}$. As an application of (2.4) as in [8], the lower bound in (2.3) was refined as

$$\sum_{1\le i<j\le n}H_{ij}\ge\binom{n}{2}\lambda^2\big[\pi+(6-2\pi)\lambda+(\pi-4)\lambda^2\big]^2\cos^2\left(\frac{\lambda}{2}\pi\right). \tag{2.5}$$

2.4. **Zhu's results.**

2.4.1. In [63], inequality (1.6) and (1.16) or inequality (1.19) and their sharpness were recovered once more by using Lemma 1.

As an application of (1.16), the upper bound in (2.3) was refined as

$$\sum_{1\le i<j\le n}H_{ij}\le4\binom{n}{2}\left[\lambda^3+\frac{\lambda\big(1-\lambda^2\big)\pi}{2}\right]^2. \tag{2.6}$$

2.4.2. In [64], by using Lemma 1, inequality (2.4) and the following two refined forms of Jordan's inequality

$$\frac{12-\pi^2}{16\pi^5}\bigl(\pi^2-4x^2\bigr)^2 \le \frac{\sin x}{x}-\frac{2}{\pi}-\frac{1}{\pi^3}\bigl(\pi^2-4x^2\bigr) \le \frac{\pi-3}{\pi^5}\bigl(\pi^2-4x^2\bigr)^2, \tag{2.7}$$

$$\frac{\sin x}{x} \le \frac{2}{\pi}+\frac{1}{\pi^3}\bigl(\pi^2-4x^2\bigr)+\frac{12-\pi^2}{\pi^3}\left(x-\frac{\pi}{2}\right)^2 \tag{2.8}$$

were established. Inequality (2.7) and the right hand side inequality in (2.8) were also applied to obtain

$$N_3(\lambda) \le \sum_{1\le i<j\le n} H_{ij} \le \min\bigl\{M_3(\lambda), M_3'(\lambda)\bigr\}, \tag{2.9}$$

where

$$\begin{aligned} N_3(\lambda) &= \binom{n}{2}\lambda^2\left[3-\lambda^2+\frac{12-\pi^2}{16}\bigl(1-\lambda^2\bigr)^2\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right),\\ M_3(\lambda) &= \binom{n}{2}\lambda^2\bigl[3-\lambda^2+(\pi-3)\bigl(1-\lambda^2\bigr)^2\bigr]^2,\\ M_3'(\lambda) &= \binom{n}{2}\lambda^2\left[3-\lambda^2+\frac{12-\pi^2}{4}(1-\lambda)^2\right]^2. \end{aligned}$$

2.5. **Jiang-Hua's result.** In [18], by Lemma 1, a refinement of Jordan's inequality

$$\frac{1}{2\pi^5}\bigl(\pi^4-16x^4\bigr) \le \frac{\sin x}{x}-\frac{2}{\pi} \le \frac{\pi-2}{\pi^5}\bigl(\pi^4-16x^4\bigr) \tag{2.10}$$

for $0<x\le\frac{\pi}{2}$ was presented. Meanwhile, L. Yang's inequality was refined as

$$\binom{n}{2}\frac{\lambda^2\bigl(5-\lambda^4\bigr)^2}{4}\cos^2\left(\frac{\lambda}{2}\pi\right) \le \sum_{1\le i<j\le n} H_{ij} \le \binom{n}{2}\lambda^4\bigl[1+2\lambda^3-\lambda^4\bigr]^2. \tag{2.11}$$

2.6. **Qi-Niu-Cao's result.** Recently, the following general refinement of Jordan's inequality was presented in [42]: For $0<x\le\frac{\pi}{2}$ and $n\in\mathbb{N}$, inequality

$$\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k\bigl(\pi^2-4x^2\bigr)^k \le \frac{\sin x}{x} \le \frac{2}{\pi}+\sum_{k=1}^{n}\beta_k\bigl(\pi^2-4x^2\bigr)^k \tag{2.12}$$

holds with the equalities if and only if $x=\frac{\pi}{2}$, where the constants

$$\alpha_k=\frac{(-1)^k}{(4\pi)^k k!}\sum_{i=1}^{k+1}\left(\frac{2}{\pi}\right)^i a_{i-1}^k\sin\left(\frac{k+i}{2}\pi\right) \tag{2.13}$$

and

$$\beta_k=\begin{cases}\dfrac{1-\frac{2}{\pi}-\sum_{i=1}^{n-1}\alpha_i\pi^{2i}}{\pi^{2n}}, & k=n\\ \alpha_k, & 1\le k<n\end{cases} \tag{2.14}$$

with

$$a_i^k=\begin{cases}(i+k-1)a_{i-1}^{k-1}+a_i^{k-1}, & 0<i\le k\\ 1, & i=0\end{cases} \tag{2.15}$$

in (2.12) are the best possible. As an application of inequality (2.12), a refinement of L. Yang's inequality [55] is obtained: For $0 \le \lambda \le 1$ and $A_i > 0$ such that $\sum_{i=1}^{n} A_i \le \pi$ for $n \in \mathbb{N}$, if $m \in \mathbb{N}$ and $n \ge 2$, then

$$L_m(n,\lambda) \le \sum_{1\le i<j\le n} H_{ij} \le R_m(n,\lambda), \tag{2.16}$$

where

$$L_m(n,\lambda) = \binom{n}{2}\lambda^2\left[2+\sum_{k=1}^{m}\alpha_k\pi^{2k+1}\left(1-\lambda^2\right)^k\right]^2\cos^2\left(\frac{\lambda}{2}\pi\right), \tag{2.17}$$

$$R_m(n,\lambda) = \binom{n}{2}\lambda^2\left[2+\sum_{k=1}^{m}\beta_k\pi^{2k+1}\left(1-\lambda^2\right)^k\right]^2, \tag{2.18}$$

and α_k and β_k are defined by (2.13) and (2.14) respectively.

Aa a direct consequence of (2.12), the following general refinements of Kober's inequality can be obtained: For $0 < x \le \frac{\pi}{2}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$, inequalities

$$\begin{aligned}\left(x-\frac{\pi}{2}\right)\left[\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k(4x)^k\left(\pi-x\right)^k\right] &\le \cos x\\ &\le \left(x-\frac{\pi}{2}\right)\left[\frac{2}{\pi}+\sum_{k=1}^{n}\beta_k(4x)^k\left(\pi-x\right)^k\right],\end{aligned} \tag{2.19}$$

which is deduced by replacing x with $x-\frac{\pi}{2}$ in (2.12), and

$$\begin{aligned}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^i\binom{k}{i}\alpha_k\pi^{2k-2i}}{2i+2}x^{2i+2} &\le 1-\cos x-\frac{x^2}{\pi}\\ &\le \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^i\binom{k}{i}\beta_k\pi^{2k-2i}}{2i+2}x^{2i+2},\end{aligned} \tag{2.20}$$

which follows from integrating (2.12) from 0 to $x \in \left[0,\frac{\pi}{2}\right]$, hold with constants α_k and β_k defined by (2.13) and (2.14) respectively.

Combining $\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}$ with (2.12) yields that if $0 < x < \frac{\pi}{2}$ and $n \in \mathbb{N}$ then

$$\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k\left(\pi^2-4x^2\right)^k \le \frac{1}{\Gamma\left(1+\frac{x}{\pi}\right)\Gamma\left(1-\frac{x}{\pi}\right)} \le \frac{2}{\pi}+\sum_{k=1}^{n}\beta_k\left(\pi^2-4x^2\right)^k. \tag{2.21}$$

Inequality (2.12) can be rearranged as

$$0 \le \frac{\sin x}{x}-\frac{2}{\pi}-\sum_{k=1}^{n}\alpha_k\left(\pi^2-4x^2\right)^k \le \sum_{k=1}^{n}(\beta_k-\alpha_k)\left(\pi^2-4x^2\right)^k \to 0$$

as $n \to \infty$, this implies that

$$\sin x = \frac{2}{\pi}x-\sum_{k=1}^{\infty}\alpha_k x\left(\pi^2-4x^2\right). \tag{2.22}$$

3. Generalizations of Jordan's inequality and L. Yang's inequality

3.1. **Zhu's generalization and application.** In [62], by using Lemma 1, the author obtained the following generalization of Jordan's inequality: If $0 < x \le r \le \frac{\pi}{2}$, then

$$\frac{\sin r}{r} + \frac{\sin r - r\cos r}{2r^3}\big(r^2 - x^2\big) \le \frac{\sin x}{x} \le \frac{\sin r}{r} + \frac{r - \sin r}{r^3}\big(r^2 - x^2\big). \tag{3.1}$$

As an application of (3.1), in virtue of (2.2), L. Yang's inequality (2.1) was sharpened and generalized as

$$\begin{aligned} 4\binom{n}{2}\left[\frac{\lambda\pi\sin r}{2r} + \frac{\sin r - r\cos r}{2r^3}\left(\frac{\lambda\pi r^2}{2} - \frac{(\lambda\pi)^3}{8}\right)\right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right) \\ \le \sum_{1\le i<j\le n} H_{ij} \le 4\binom{n}{2}\left[\frac{\lambda\pi\sin r}{2r} + \frac{r - \sin r}{r^3}\left(\frac{\lambda\pi r^2}{2} - \frac{(\lambda\pi)^3}{8}\right)\right]^2. \end{aligned} \tag{3.2}$$

3.2. **Wu-Debnath's generalizations and applications.** In [52], utilizing Lemma 1, the following sharp generalizations of Jordan's inequality

$$\begin{aligned} \max\left\{\frac{3}{2}\varphi_1(\theta)\left(1 - \frac{x}{\theta}\right)^2, \frac{3}{8}\varphi_2(\theta)\left(1 - \frac{x^2}{\theta^2}\right)^2\right\} \\ \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \frac{1}{2}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\left(1 - \frac{x^2}{\theta^2}\right) \\ \le \min\left\{\frac{3}{2}\varphi_2(\theta)\left(1 - \frac{x}{\theta}\right)^2, \frac{3}{2}\varphi_1(\theta)\left(1 - \frac{x^2}{\theta^2}\right)^2\right\} \end{aligned} \tag{3.3}$$

for $0 < x \le \theta$ and $\theta \in (0, \pi]$ was established, where

$$\varphi_1(\theta) = \frac{2}{3} + \frac{\cos\theta}{3} - \frac{\sin\theta}{\theta} \quad \text{and} \quad \varphi_2(\theta) = \frac{\sin\theta}{\theta} - \frac{1}{3}\theta\sin\theta - \cos\theta. \tag{3.4}$$

The equalities in (3.3) hold if and only if $x = \theta$ and the coefficients of the factors $\big(1 - \frac{x}{\theta}\big)^2$ and $\big(1 - \frac{x^2}{\theta^2}\big)^2$ are the best possible.

If taking $\theta = \frac{\pi}{2}$ then inequalities (2.7) and (2.8) can be deduced from (3.3).

Integrating on both sides of (3.3) yields

$$\begin{aligned} \max\left\{\frac{5\sin\theta - \theta\cos\theta + 2\theta}{6}, \frac{23\sin\theta - 8\theta\cos\theta - \theta^2\sin\theta}{15}\right\} < \int_0^\theta \frac{\sin x}{x}\,\mathrm{d}x \\ < \min\left\{\frac{11\sin\theta - 5\theta\cos\theta - \theta^2\sin\theta}{6}, \frac{8\sin\theta - \theta\cos\theta + 8\theta}{15}\right\}. \end{aligned} \tag{3.5}$$

If taking $\theta = \frac{\pi}{2}$ in (3.5), then

$$\frac{92 - \pi^2}{60} < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{8 + 4\pi}{15} \tag{3.6}$$

which is better than the first one in (1.20).

As another application of (3.3), a generalization of L. Yang's inequality (2.1) was obtained: If $A_i > 0$ for $1 \le i \le n$ and $n \ge 2$ such that $\sum_{i=1}^n A_i \le \theta \in [0, \pi]$, then

$$
\begin{aligned}
\max\{N_1(\theta), N_2(\theta)\} &\le \binom{n}{2}\sin^2\theta \\
&\le (n-1)\sum_{k=1}^{n}\cos^2 A_k - 2\cos\theta \sum_{1\le i<j\le n}\cos A_i\cos A_j \\
&\le 4\binom{n}{2}\sin^2\frac{\theta}{2} \le \min\{M_1(\theta), M_2(\theta)\}, \quad (3.7)
\end{aligned}
$$

where

$$
N_1(\theta) = \binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+(\pi-3)\left(1-\frac{\theta}{\pi}\right)^2\right]^2\left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2, \quad (3.8)
$$

$$
N_2(\theta) = \binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+\frac{12-\pi^2}{16}\left(1-\frac{\theta^2}{\pi^2}\right)^2\right]^2\left(\frac{\theta}{\pi}\cos\frac{\theta}{2}\right)^2, \quad (3.9)
$$

$$
M_1(\theta) = \binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+\frac{12-\pi^2}{4}\left(1-\frac{\theta}{\pi}\right)^2\right]^2\left(\frac{\theta}{\pi}\right)^2, \quad (3.10)
$$

$$
M_2(\theta) = \binom{n}{2}\left[3-\frac{\theta^2}{\pi^2}+(\pi-3)\left(1-\frac{\theta^2}{\pi^2}\right)^2\right]^2\left(\frac{\theta}{\pi}\right)^2. \quad (3.11)
$$

If substituting A_i by λA_i and θ by $\lambda\pi$ in (3.7), then inequalities (2.4) and (2.9) can be deduced.

In [53], as a generalization of inequality (3.3), the following sharp inequality

$$
\begin{aligned}
\frac{1}{2\tau^2}&\left[(1+\lambda)\left(\frac{\sin\theta}{\theta}-\cos\theta\right)-\theta\sin\theta\right]\left(1-\frac{x^\tau}{\theta^\tau}\right)^2 \\
&\le \frac{\sin x}{x}-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\left(1-\frac{x^\lambda}{\theta^\lambda}\right) \\
&\le \left[1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\right]\left(1-\frac{x^\tau}{\theta^\tau}\right)^2 \quad (3.12)
\end{aligned}
$$

was obtained for $0 < x \le \theta \in \left(0, \frac{\pi}{2}\right]$, $\tau \ge 2$ and $\tau \le \lambda \le 2\tau$ by Lemma 1. The equalities in (3.12) holds if and only if $x=\theta$. The coefficients of the term $\left(1-\frac{x^\tau}{\theta^\tau}\right)^2$ are the best possible. If $1 \le \tau \le \frac{5}{3}$ and either $\lambda \ne 0$ or $\lambda \ge 2\tau$ then inequality (3.12) is reversed. Specially, when $\theta = \frac{\pi}{2}$, inequality (3.12) becomes

$$
\begin{aligned}
\frac{4\lambda+4-\pi^2}{4\tau^2\pi^{2\tau+1}}\left(\pi^\tau-2^\tau x^\tau\right)^2 &\le \frac{\sin x}{x}-\frac{2}{\pi}-\frac{2}{\lambda\pi^{\lambda+1}}\left(\pi^\lambda-2^\lambda x^\lambda\right) \\
&\le \frac{\lambda\pi-2\lambda-2}{\lambda\pi^{2\tau+1}}\left(\pi^\tau-2^\tau x^\tau\right)^2 \quad (3.13)
\end{aligned}
$$

for $0 < x \le \frac{\pi}{2}$, $\tau \ge 2$ and $\tau \le \lambda \le 2\tau$. If $1 \le \tau \le \frac{5}{3}$ and either $\lambda \ne 0$ or $\lambda \ge 2\tau$ then inequality (3.13) is reversed.

If taking $(\tau, \lambda) = (2, 2)$ and $(\tau, \lambda) = (1, 2)$, then inequalities (2.4), (2.7) and (2.8) are deduced.

If $\lambda \ge 2$ and $A_i \ge 0$ with $\sum_{i=1}^{n} A_i \le \theta \in [0,\pi]$ for $n \ge 2$, then the following generalization of L. Yang's inequality was obtained in [53] by using inequality (3.12):

$$\max\{K_1(\lambda,\theta),K_2(\lambda,\theta)\} \le (n-1)\sum_{k=1}^{n}\cos^2 A_k - 2\cos\theta\sum_{1\le i<j\le n}\cos A_i\cos A_j \le \min\{Q_1(\lambda,\theta),Q_2(\lambda,\theta)\}, \quad (3.14)$$

where

$$K_1(\lambda,\theta) = \binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{\lambda\pi-2\lambda-2}{2}\left(1-\frac{\theta}{\pi}\right)^2\right]\frac{2\theta}{\lambda\pi}\cos\frac{\theta}{2}\right\}^2, \quad (3.15)$$

$$K_2(\lambda,\theta) = \binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{4\lambda\pi+4-\pi^2}{8\lambda}\left(1-\frac{\theta^\lambda}{\pi^\lambda}\right)^2\right]\frac{2\theta}{\lambda\pi}\cos\frac{\theta}{2}\right\}^2, \quad (3.16)$$

$$Q_1(\lambda,\theta) = \binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{4\lambda+4\lambda^2-\lambda\pi^2}{8}\left(1-\frac{\theta}{\pi}\right)^2\right]\frac{2\theta}{\lambda\pi}\right\}^2, \quad (3.17)$$

$$Q_2(\lambda,\theta) = \binom{n}{2}\left\{\left[\lambda+1-\frac{\theta^\lambda}{\pi^\lambda}+\frac{\lambda\pi-2\lambda-2}{2}\left(1-\frac{\theta^\lambda}{\pi^\lambda}\right)^2\right]\frac{2\theta}{\lambda\pi}\right\}^2. \quad (3.18)$$

Note that inequalities (2.5), (2.9) and (3.7) can be deduced from (3.14).

4. Wilker's inequality and its proofs

In [48], J. B. Wilker proposed that there exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x} > 2 + cx^3\tan x \quad (4.1)$$

for $0 < x < \frac{\pi}{2}$.

In [46], it was proved that

$$2+\frac{8}{45}x^3\tan x > \left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x} > 2+\left(\frac{2}{\pi}\right)^4 x^3\tan x. \quad (4.2)$$

The constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ in the inequality (4.2) are the best possible.

In [11, 12, 14, 24, 58], many proofs of Wilker's inequality (4.2) were given.

In [29], a new proof of inequality (4.2) were provided by using Lemma 1 and compared with [14].

The weaker form of inequality (4.2)

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x} > 2. \quad (4.3)$$

was also proved in [6, 24, 47, 61].

In [17, 54] two lower bounds of $\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2$ were presented, but these lower bounds are weaker than $\left(\frac{2}{\pi}\right)^4 x^3\tan x$ in (4.2).

It is noted that one of the two open problems posed in [54] may be interesting.

5. Applications of a method of auxiliary functions

The aim of this section is to summarize some applications of a method of auxiliary functions, used in [31, 32, 34, 37, 39], including estimation of some complete elliptic integrals and construction of inequality for the exponential function e^x.

The complete elliptic integrals are classed into three kinds, they are defined for $0<k<1$ as and denoted by

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta, \tag{5.1}$$

$$F(k)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{5.2}$$

$$\varPi(k,h)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}. \tag{5.3}$$

5.1. In [43], it was posed that

$$\frac{\pi}{6}<\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x<\frac{\pi\sqrt{2}}{8}. \tag{5.4}$$

In [10], inequality (5.4) was verified by using $4-x^2>4-x^2-x^3>4-2x^2$.

In [38], by considering monotonicity and convexity of

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^4+\alpha x^3(1-x) \tag{5.5}$$

in $(0,1)$ for undetermined constant $\alpha\ge 0$, inequality

$$\frac{1}{\sqrt{4-x^2-x^3}}\ge\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^4+\left(\frac{11\sqrt{2}}{8}-2\right)(1-x)x^3 \tag{5.6}$$

for $x\in[0,1]$ was established, and then the lower bound in (5.4) was improved to

$$\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x>\frac{3}{10}+\frac{27\sqrt{2}}{160}. \tag{5.7}$$

It was also remarked in [38] that if discussing the auxiliary functions

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^2+\beta(1-x)x^2 \tag{5.8}$$

and

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^4+\theta(1-x^3)x \tag{5.9}$$

in $(0,1)$, then inequalities

$$\frac{1}{\sqrt{4-x^2-x^3}}\ge\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2+\left(\frac{3\sqrt{2}}{8}-1\right)(1-x)x^2 \tag{5.10}$$

and

$$\frac{1}{\sqrt{4-x^2-x^3}}\ge\frac{1}{2}+\frac{\sqrt{2}-1}{2}x^4+\left(\frac{2}{3}-\frac{11\sqrt{2}}{24}\right)(x^3-1)x \tag{5.11}$$

holds, and then, by integrating on both sides, the lower bound in (5.4) was improved to

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x > \frac{1}{4}+\frac{19\sqrt{2}}{96} \tag{5.12}$$

and

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x > \frac{1}{5}+\frac{19\sqrt{2}}{80}. \tag{5.13}$$

Numerical computation shows that the lower bound in (5.7) is better than those in (5.12) and (5.13).

In [56], by direct proving inequality (5.6) and

$$\frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2+\frac{5-4\sqrt{2}}{8}x^2(1-x)\biggl(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\biggr), \tag{5.14}$$

inequality (5.7) and an improved upper bound in (5.4)

$$\int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\,\mathrm{d}x < \frac{79}{192}+\frac{\sqrt{2}}{10} \tag{5.15}$$

were obtained.

In [35], by considering an auxiliary function

$$\frac{1}{\sqrt{4-x^2-x^3}}-\frac{1}{2}+\frac{1-\sqrt{2}}{2}x^2+\alpha x^2(1-x)\biggl(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\biggr) \tag{5.16}$$

on $[0,1]$, the sharpness of inequality (5.14) and the following sharp inequality

$$\frac{1}{\sqrt{4-x^2-x^3}} \ge \frac{1}{2}+\frac{\sqrt{2}-1}{2}x^2-\frac{1137\bigl(4\sqrt{2}-5\bigr)}{64\bigl(64-39\sqrt{2}\bigr)}(1-x)\biggl(\frac{8\sqrt{2}-9}{8\sqrt{2}-10}+x\biggr) \tag{5.17}$$

were presented, and then inequality (5.15) was obtained by integrating on both sides of (5.14).

5.2. In [13], by discussing

$$\sqrt{1+k^2\cos^2 t}-\sqrt{1+k^2}+\frac{4}{\pi^2}\Bigl(\sqrt{1+k^2}-1\Bigr)t^2+\theta\Bigl(\frac{\pi}{2}-t\Bigr)t \tag{5.18}$$

or

$$\sqrt{1+k^2\cos^2 t}-\sqrt{1+k^2}+\frac{2}{\pi}\Bigl(\sqrt{1+k^2}-1\Bigr)t+\beta\Bigl(\frac{\pi}{2}-t\Bigr)t \tag{5.19}$$

on $\bigl[0,\frac{\pi}{2}\bigr]$, inequality

$$-\frac{8}{\pi^2}\Bigl(\sqrt{1+k^2}-1\Bigr)t\Bigl(\frac{\pi}{2}-t\Bigr) \le \sqrt{1+k^2\cos^2 t}-\biggl[\sqrt{1+k^2}-\frac{4}{\pi^2}\Bigl(\sqrt{1+k^2}-1\Bigr)t^2\biggr] \le 0 \tag{5.20}$$

for $t\in\bigl[0,\frac{\pi}{2}\bigr]$ was obtained, where $k^2=\frac{b^2}{a^2}-1$ and $a,b>0$. Integrating (5.20) yields

$$\frac{\pi}{6}(2a+b) < \int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,\mathrm{d}t \le \frac{\pi}{6}(a+2b). \tag{5.21}$$

When $b \geq 7a$, the right hand side of inequality (5.21) is stronger than a well known result

$$\frac{\pi}{4}(a+b) \leq \int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,\mathrm{d}t \leq \frac{\pi}{4}\sqrt{2(a^2+b^2)} \tag{5.22}$$

which can be obtained by some properties of definite integral.

5.3. In [15, 31], by considering the auxiliary function

$$e^x - S_n(x) - \alpha_n x^{n+1} + \theta(b-x)x^{n+1} \tag{5.23}$$

for $0 \leq x \leq b \in (0,\infty)$, where $\alpha_{-1} = e^b$ and $\alpha_n = \frac{1}{b}\bigl(\alpha_{n-1} - \frac{1}{n!}\bigr)$, the following inequalities of the reminder $R_n(x) = e^x - \sum_{k=0}^n \frac{x^k}{k!}$ for $n \geq 0$ and $x \in [0,\infty)$ were established:

$$\frac{n+2-(n+1)x}{(n+2)!}x^{n+1}e^x \leq R_n(x) \leq \frac{n+1+e^x}{(n+2)!}x^{n+1} \leq \frac{e^x}{(n+1)!}x^{n+1}, \tag{5.24}$$

$$\frac{(n+2)!}{(n-k+2)!}R_n(x) \leq x^k R_{n-k}(x) + \frac{k}{(n-k+2)!}x^{n+1}, \quad 0 \leq k \leq n \tag{5.25}$$

and, for $n \geq k \geq 1$,

$$x^k R_{n-k}(x) \leq \frac{kx^{n+1}e^x}{(n+1)(n-k+2)!} - \frac{n! - (n-k+2)(n+1)!}{(n-k+2)!}R_n(x). \tag{5.26}$$

5.4. By the way, some other estimates for complete elliptic integrals obtained by using Tchebycheff's integral inequality in [40] are mentioned below.

$$\frac{\pi \arcsin k}{2k} < F(k) < \frac{\pi \ln\bigl(\frac{1+k}{1-k}\bigr)}{4k}; \tag{5.27}$$

$$E(k) < \frac{16-4k^2-3k^4}{4(4+k^2)}F(k); \tag{5.28}$$

$$F(k) < \Bigl(1+\frac{h}{2}\Bigr)\mathit{\Pi}(k,h), \quad -1<h<0 \quad \text{or} \quad h > \frac{k^2}{2-3k^2} > 0; \tag{5.29}$$

$$\mathit{\Pi}(k,h)\cdot E(k) > \frac{\pi^2}{4\sqrt{1+h}}, \quad -2 < 2h < k^2; \tag{5.30}$$

$$E(k) \geq \frac{16-28k^2+9k^4}{4(4-5k^2)}F(k), \quad k^2 \leq \frac{2}{3}. \tag{5.31}$$

For $0 < 2h < k^2$, inequality (5.29) is reversed. For $h > \frac{k^2}{2-3k^2} > 0$, inequality (5.30) is reversed. As concrete examples the following estimates of the complete elliptic integrals can be deduced:

$$\frac{\pi^2}{4\sqrt{2}} < \int_0^{\pi/2}\Bigl(1-\frac{\sin^2 x}{2}\Bigr)^{-1/2}\mathrm{d}x < \frac{\pi\ln(1+\sqrt{2}\,)}{\sqrt{2}}, \tag{5.32}$$

$$\int_0^{\pi/2}\biggl(1+\frac{\cos x}{2}\biggr)^{-1}\mathrm{d}x < \frac{\pi(\ln 3-\ln 2)}{2}, \tag{5.33}$$

$$\int_0^{\pi/2}\biggl(1-\frac{\sin x}{2}\biggr)^{-1}\mathrm{d}x = \int_{\pi/2}^{\pi}\biggl(1+\frac{\cos x}{2}\biggr)^{-1}\mathrm{d}x > \frac{\pi\ln 2}{2}. \tag{5.34}$$

These results are better than those in [22, p. 607].

References

[1] U. Abel and D. Caccia, *A sharpening of Jordan's inequality*, Amer. Math. Monthly **93** (1986), no. 7, 568–569.

[2] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 4th printing, with corrections, Applied Mathematics Series **55**, National Bureau of Standards, Washington, 1965.

[3] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasi-conformal Maps*, John Wiley & Sons, New York, 1997.

[4] Á. Baricz, *Redheffer's inequality for Bessel function*, submitted.

[5] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.

[6] Ch.-P. Chen and F. Qi, *On two new proofs of Wilker's inequality*, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) **5** (2002), no. 4, 38–39.

[7] Ch.-P. Chen, J.-W. Zhao, and F. Qi, *Three inequalities involving hyperbolically trigonometric functions*, Octogon Math. Mag. **12** (2004), no. 2, 592–596. RGMIA Res. Rep. Coll. **6** (2003), no. 3, Art. 4, 437–443. Available online at `http://rgmia.vu.edu.au/v6n3.html`.

[8] L. Debnath and Ch.-J. Zhao, *New strengthened Jordan's inequality and its applications*, Appl. Math. Lett. **16** (2003), no. 4, 557–560.

[9] Y.-F. Feng (Feng Yuefeng), *Proof Without Words: Jordan's Inequality* $\frac{2x}{\pi} \le \sin x \le x$, $0 \le x \le \frac{\pi}{2}$, Math. Mag. **69** (1996), 126.

[10] R. H. Garstang, *Bounds on an elliptic integral*, Amer. Math. Monthly **94** (1987), no. 6, 556–557.

[11] Bai-Ni Guo, Wei Li and Feng Qi, *Proofs of Wilker's inequalities involving trigonometric functions*, Inequality Theory and Applications, Volume **2** (Chinju/Masan, 2001), 109–112, Nova Science Publishers, Hauppauge, NY, 2003.

[12] B.-N. Guo, W. Li, B.-M. Qiao and F. Qi, *On new proofs of inequalities involving trigonometric functions*, RGMIA Res. Rep. Coll. **3** (2000), no. 1, Art. 15, 167–170. Available online at `http://rgmia.vu.edu.au/v3n1.html`.

[13] B.-N. Guo, F. Qi and Sh.-J. Jing, *Improvement for the upper bound of a class of elliptic integral*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **14** (1995), no. 6, 125–128. (Chinese)

[14] B.-N. Guo, B.-M. Qiao, F. Qi and W. Li, *On new proofs of Wilker's inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), no. 1, 19–22.

[15] Q.-D. Hao, *On construction of several sharp inequalities for the exponential function* e^x, Kuàng Yè (Mining) (1995), no. 1, 39–42. (Chinese)

[16] Q.-D. Hao and B.-N. Guo, *A method of finding extremums of composite functions of trigonometric functions*, Kuàng Yè (Mining) (1993), no. 4, 80–83. (Chinese)

[17] W.-D. Jiang and Y. Hua, *Note on Wilker's inequality and Huygens's inequality*, submitted.

[18] W.-D. Jiang and Y. Hua, *Sharpening of Jordan's inequality and its applications*, Bùděngshì Yānjiū Tōngxùn (Communications in Studies on Inequalities) **12** (2005), no. 3, 288–290.

[19] G. Klambauer, *Problems and Propositions in Analysis*, Marcel Dekker, New York and Basel, 1979.

[20] H. Kober, *Approximation by integral functions in the complex domain*, Trans. Amer. Math. Soc. **56** (1944), no. 1, 7–31.

[21] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), Hunan Education Press, Changsha, China, June 1989. (Chinese)

[22] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd ed., Hunan Education Press, Changsha, China, May 1993. (Chinese)

[23] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shandong Science and Technology Press, Jinan City, Shandong Province, China, October 2004. (Chinese)

[24] A.-Q. Liu, G. Wang, and W. Li, *New proofs of Wilkers inequalities involving trigonometric functions*, Jiāozuò Gōng Xuéyuàn Xuébào (Journal of Jiaozuo Institute of Technology (Natural Science)) **21** (2002), no. 5, 401–403.

[25] Q.-M. Luo, Z.-L. Wei and F. Qi, *Lower and upper bounds of* $\zeta(3)$, Adv. Stud. Contemp. Math. (Kyungshang) **6** (2003), no. 1, 47–51. RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 7, 565–569. Available online at `http://rgmia.vu.edu.au/v4n4.html`.

[26] A. McD. Mercer, *Problem E 2952*, Amer. Math. Monthly **89** (1982), no. 6, 424.

[27] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, 1970.
[28] A. Y. Özban, *A new refined form of Jordan's inequality and its applications*, Appl. Math. Lett. **19** (2006), no. 2, 155–160.
[29] I. Pinelis, *L'Hospital rules for monotonicity and the Wilker-Anglesio inequality*, Amer. Math. Monthly **111** (2004), 905–909.
[30] J. Prestin, *Trigonometric interpolation in Hölder spaces*, J. Approx. Theory **53** (1988), no. 2, 145–154.
[31] F. Qi, *A method of constructing inequalities about* e^x, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 16–23.
[32] F. Qi, *Extensions and sharpenings of Jordan's and Kober's inequality*, Gōngkē Shùxué (Journal of Mathematics for Technology) **12** (1996), no. 4, 98–102. (Chinese)
[33] F. Qi, L.-H. Cui, and S.-L. Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (1999), no. 4, 517–528.
[34] F. Qi and B.-N. Guo, *Extensions and sharpenings of the noted Kober's inequality*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **12** (1993), no. 4, 101–103. (Chinese)
[35] F. Qi and B.-N. Guo, *Estimate for upper bound of an elliptic integral*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory) **26** (1996), no. 3, 285–288. (Chinese)
[36] F. Qi and B.-N. Guo, *Lower bound of the first eigenvalue for the Laplace operator on compact Riemannian manifold*, Chinese Quart. J. Math. **8** (1993), no. 2, 40–49.
[37] F. Qi and B.-N. Guo, *On generalizations of Jordan's inequality*, Méitàn Gāoděng Jiàoyù (Coal Higher Education), supplement, November/1993, 32–33. (Chinese)
[38] F. Qi and B.-N. Guo, *The estimation of inferior bound for an ellipse integral*, Gōngkē Shùxué (Journal of Mathematics for Technology) **10** (1994), no. 1, 87–90. (Chinese)
[39] F. Qi and Q.-D. Hao, *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (1998), no. 3, 116–120.
[40] F. Qi and Zh. Huang, *Inequalities of the complete elliptic integrals*, Tamkang J. Math. **29** (1998), no. 3, 165–169.
[41] F. Qi, H.-Ch. Li, B.-N. Guo and Q.-M. Luo, *Inequalities and estimates of the eigenvalue for Laplace operator*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **13** (1994), no. 3, 89–95. (Chinese)
[42] F. Qi, D.-W. Niu, and J. Cao, *A general refinement of Jordan's inequality and a refinement of L. Yang's inequality*, submitted.
[43] Th. M. Rassias, *Problem E 3111*, Amer. Math. Monthly **92** (1985), no. 9, 665.
[44] R. Redheffer, *Problem 5642*, Amer. Math. Monthly **75** (1968), no. 10, 1125.
[45] R. Redheffer, *Correction*, Amer. Math. Monthly **76** (1969), no. 4, 422.
[46] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), no. 3, 264–267.
[47] J.-Sh. Sun, *Two simple proof of Wiker's inequality involving trigonometric functions*, Gāoděng Shùxué Yánjiū (Studies in College Mathematics) **7** (2004), no. 4, 43.
[48] J. B. Wilker, *Problem E 3306*, Amer. Math. Monthly **96** (1989), no. 1, 55.
[49] J. P. Williams, *A delightful inequality*, Amer. Math. Monthly **76** (1969), no. 10, 1153–1154.
[50] Sh.-H. Wu, *On generalizations and refinements of Jordan type inequality*, Octogon Math. Mag. **12** (2004), no. 1, 267–272.
[51] Sh.-H. Wu, *On generalizations and refinements of Jordan type inequality*, RGMIA Res. Rep. Coll. **7** (2004), Suppl., Art. 2. Available online at `http://rgmia.vu.edu.au/v7(E).html`
[52] Sh-H. Wu and L. Debnath, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality*, Appl. Math. Lett. **19** (2006), in press.
[53] Sh-H. Wu and L. Debnath, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality, II*, Appl. Math. Lett. **19** (2006), in press.
[54] Sh.-Ch. Yang, *A sharpening of Wilker inequality*, Ābà Shīfàn Gāoděng Zhuānkē Xuéxiào Xuébào (Journal of Aba Teachers College) (2003), no. 3, 104–105.
[55] L. Yang, *Zhí Fēnbù Lǐlùn Jíqí Xīn Yánjiū* (*The Theory of Distribution of Values of Functions and Recent Researches*), Science Press, Beijing, 1982.(Chinese)
[56] L.-Q. Yu, F. Qi and B.-N. Guo, *Estimates for upper and lower bounds of a complete elliptic integral*, Kuàng Yè (Mining) (1995), no. 1, 35–38. (Chinese)

[57] X.-H. Zhang, G.-D. Wang, Y.-M. Chu, *Extensions and sharpenings of Jordan's and Kober's inequalities*, J. Inequal. Pure Appl. Math. **7** (2006), no. 2, Art. 63. Avaialble online at `http://jipam.vu.edu.au/article.php?sid=680`.

[58] L. Zhang and L. Zhu, *A new elementary proof of Wilker's inequalities*, submitted.

[59] Ch.-J. Zhao, *Generalization and strengthening of the Yang Le inequality*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory) **30** (2000), no. 4, 493–497 (Chinese).

[60] Ch.-J. Zhao and L. Debnath, *On generalizations of L. Yang's inequality*, J. Inequal. Pure Appl. Math. **3** (2002), no. 4, Art. 56. Available online at `http://jipam.vu.edu.au/article.php?sid=208`.

[61] L. Zhu, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. **8** (2005), no. 4, 749–750.

[62] L. Zhu, *Sharpening of Jordan's inequalities and its applications*, Math. Inequal. Appl. **9** (2006), no. 1, 103–106.

[63] L. Zhu, *Sharpening Jordan's inequality and the Yang Le inequality*, Appl. Math. Lett. **19** (2006), no. 3, 240–243.

[64] L. Zhu, *Sharpening Jordan's inequality and the Yang Le inequality, II*, Appl. Math. Lett. **19** (2006), no. 9, 990–994.

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