

**FOUR LOGARITHMICALLY COMPLETELY MONOTONIC
FUNCTIONS INVOLVING GAMMA FUNCTION AND
ORIGINATING FROM PROBLEMS OF TRAFFIC FLOW**

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ABSTRACT. In this paper, two classes of functions, involving a parameter and the Euler gamma function, and two functions, involving the Euler gamma function, are verified to be logarithmically completely monotonic in $(-\frac{1}{2}, \infty)$ or $(0, \infty)$ and an inequality involving the Euler gamma function, due to J. Wendel, is refined partially.

1. INTRODUCTION

The Kershaw's inequality in [9] states that the double inequality

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \quad (1)$$

holds for $0 < s < 1$ and $x \geq 1$, where Γ denotes the classical Euler gamma function and $\psi = \frac{\Gamma'}{\Gamma}$, the logarithmic derivative of Γ . If taking $s = \frac{1}{2}$ in (1), then

$$\sqrt{x + \frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{x + \frac{\sqrt{3}-1}{2}}. \quad (2)$$

Let s and t be nonnegative numbers and $\alpha = \min\{s, t\}$. For $x \in (-\alpha, \infty)$, define

$$z_{s,t}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s = t. \end{cases} \quad (3)$$

In order to establish the best bounds in Kershaw's inequality (1), among other things, the papers [4, 6, 15, 19] established the following monotonicity and convexity property of $z_{s,t}(x)$: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$. This result was further generalized in the papers [12, 13].

In [5, p. 123] and [10], while ones studied certain problems of traffic flow, the following double inequality was obtained for $n \in \mathbb{N}$:

$$2\Gamma\left(n + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(n+1) \leq 2^n\Gamma\left(n + \frac{1}{2}\right), \quad (4)$$

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which can be rearranged for $n > 1$ as

$$1 \leq \left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)} \right]^{1/(n-1)} \leq 2. \quad (5)$$

In [23], by using the following double inequality due to J. Wendel in [27]:

$$\left(\frac{x}{x+a} \right)^{1-a} \leq \frac{\Gamma(x+a)}{x^a\Gamma(x)} \leq 1 \quad (6)$$

for $0 < a < 1$ and $x > 0$, inequality (4) was extended and refined as

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \sqrt{x + \frac{1}{2}} \quad (7)$$

for $x > 0$.

It is clear that the double inequality (7) is weaker than (2).

Recall [14, 24, 28] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \leq (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and $n \geq 0$. The set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well known Bernstein's Theorem [28, p. 161] states that $f \in \mathcal{C}[(0, \infty)]$ if and only if $f(x) = \int_0^\infty e^{-xs} d\mu(s)$, where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. This expresses that $f \in \mathcal{C}[(0, \infty)]$ if and only if f is a Laplace transform of the measure μ .

Recall [2, 7, 14, 16, 17, 20, 21] also that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I . The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. In [3, Theorem 1.1] and [7, 21] it is pointed out that the logarithmically completely monotonic functions on $(0, \infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [8, Theorem 4.4].

It was proved in [3, 14, 18, 20, 21, 24] that $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely. Stimulated by the papers [17, 20], among other things, it was further revealed in [3] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)]$, where \mathcal{S} denotes the set of Stieltjes transforms.

Let $\gamma = 0.57721566 \dots$ be Euler-Mascheroni's constant. For $x \in (-\frac{1}{2}, \infty)$, let

$$g(x) = \begin{cases} \left[\frac{\Gamma(1/2)\Gamma(x+1)}{2\Gamma(x+1/2)} \right]^{1/(x-1)}, & x \neq 1, \\ \exp \left[1 - \gamma - \psi \left(\frac{3}{2} \right) \right], & x = 1. \end{cases} \quad (8)$$

The first aim of this paper is to show the logarithmically complete monotonicity of $g(x)$. The first main result of this paper is as follows.

Theorem 1. *Let $g(x)$ be the function defined by (8). Then $g(x) \in \mathcal{L}[(-\frac{1}{2}, \infty)]$ with $\lim_{x \rightarrow -\frac{1}{2}^+} g(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = 1$.*

Remark 1. From the decreasingly monotonic property of $g(x)$ and $\lim_{x \rightarrow \infty} g(x) = 1$ in Theorem 1 and $g(1) = \exp \left[1 - \gamma - \psi \left(\frac{3}{2} \right) \right]$, it is obtained that

$$2\Gamma \left(x + \frac{1}{2} \right) \leq \Gamma \left(\frac{1}{2} \right) \Gamma(x+1) \leq 2\Gamma \left(x + \frac{1}{2} \right) \exp \left\{ (x-1) \left[1 - \gamma - \psi \left(\frac{3}{2} \right) \right] \right\} \quad (9)$$

for $x \in [1, \infty)$. From $g(0) = 2$, $\lim_{x \rightarrow \infty} g(x) = 1$ and the decreasingly monotonicity of $g(x)$, it is also revealed that

$$2\Gamma\left(x + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(x+1) \leq 2^x\Gamma\left(x + \frac{1}{2}\right) \quad (10)$$

for $x \in (0, \infty)$.

Inequalities (9) and (10) extend (4) and (5) and the right hand side inequality of (9) refines the right hand side inequality of (4) and (5). Therefore, it can be said that Theorem 1 generalizes, extends, and refines the double inequalities (4) and (5).

By the way, numerical calculation shows

$$\sqrt{1 + \frac{\sqrt{3}-1}{2}} = 1.168\dots > \frac{2}{\Gamma(1/2)} \exp\left\{(1-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} = 1.128\dots$$

and

$$\sqrt{2 + \frac{\sqrt{3}-1}{2}} = 1.538\dots < \frac{2}{\Gamma(1/2)} \exp\left\{(2-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} = 1.660\dots,$$

hence, the right hand side inequality of (2) and the following inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \frac{2}{\Gamma(1/2)} \exp\left\{(x-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} \quad (11)$$

for $x \in [1, \infty)$, which is deduced from the right hand side inequality of (9), are not included each other. Similarly, it is easy to show that inequality (11) and the right hand side inequality in (7) are also not included each other in $x \in [1, \infty)$.

For $x > 0$ and $a > 0$, let

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)}. \quad (12)$$

The second aim of this paper is to prove the logarithmically complete monotonicity of $h_a(x)$. The second main result of this paper is as follows.

Theorem 2. *Let $h_a(x)$ be the function defined by (12). Then*

- (1) $\lim_{x \rightarrow 0^+} h_a(x) = \frac{\Gamma(a+1)}{a^a}$ and $\lim_{x \rightarrow \infty} h_a(x) = 1$ for any $a > 0$,
- (2) $h_a(x) \in \mathcal{L}[(0, \infty)]$ if $0 < a < 1$,
- (3) $[h_a(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ if $a > 1$.

Let $\gamma = 0.57721566\dots$ be the Euler-Mascheroni's constant. For $x \in (0, \infty)$, define

$$p(x) = \begin{cases} \left[\frac{x^x}{\Gamma(x+1)}\right]^{1/(1-x)}, & x \neq 1, \\ e^{-\gamma}, & x = 1. \end{cases} \quad (13)$$

The third aim of this paper is to present the logarithmically complete monotonicity of $p(x)$. The third main result of this paper is as follows.

Theorem 3. *Let $p(x)$ be the function defined by (13). Then $p(x) \in \mathcal{L}[(0, \infty)]$ with $\lim_{x \rightarrow 0^+} p(x) = 1$ and $\lim_{x \rightarrow \infty} p(x) = \frac{1}{e}$.*

For $x \in (0, \infty)$ and $a \in (0, \infty)$, let

$$f_a(x) = \frac{\Gamma(x+a)}{x^a \Gamma(x)}. \quad (14)$$

The fourth aim of this paper is to consider the logarithmically complete monotonicity of $f_a(x)$. The fourth main result of this paper is as follows.

Theorem 4. *Let $f_a(x)$ be the function defined by (14). Then*

- (1) $\lim_{x \rightarrow \infty} f_a(x) = 1$ for any $a \in (0, \infty)$,
- (2) $f_a(x) \in \mathcal{L}[(0, \infty)]$ and $\lim_{x \rightarrow 0+} f_a(x) = \infty$ if $a > 1$,
- (3) $[f_a(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ and $\lim_{x \rightarrow 0+} f_a(x) = 0$ if $0 < a < 1$.

As a straightforward consequence of Theorem 2 and Theorem 4, the following refinement of inequality (6) is established.

Theorem 5. *Let $x \in (0, \infty)$. If $0 < a < 1$, then*

$$\begin{aligned} \left(\frac{x}{x+a}\right)^{1-a} &< \frac{\Gamma(x+a)}{x^a \Gamma(x)} \\ &< \begin{cases} \frac{\Gamma(a+1)}{a^a} \left(\frac{x}{x+a}\right)^{1-a} \leq 1, & 0 < x \leq \frac{ap(a)}{1-p(a)}, \\ 1, & \frac{ap(a)}{1-p(a)} < x < \infty, \end{cases} \end{aligned} \quad (15)$$

where $p(x)$ is defined by (13). If $a > 1$, the reversed inequality of (15) holds.

Remark 2. The graph of the function $\frac{ap(a)}{1-p(a)}$, pictured by MATHEMATICA 5.2, shows that it is an increasing function from $(0, \infty)$ to $(0, \infty)$.

2. PROOFS OF THEOREMS

It is well-known that, for $x > 0$ and $\omega > 0$,

$$\frac{1}{x^\omega} = \frac{1}{\Gamma(\omega)} \int_0^\infty t^{\omega-1} e^{-xt} dt, \quad (16)$$

and that, for $k \in \mathbb{N}$ and $x > 0$,

$$\psi(x) = \ln x + \int_0^\infty \left(\frac{1}{u} - \frac{1}{1-e^{-u}}\right) e^{-xu} du, \quad (17)$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k}{1-e^{-t}} e^{-xt} dt. \quad (18)$$

Moreover, as $x \rightarrow \infty$, the following asymptotic formula holds:

$$x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \quad (19)$$

where a and b are two constants.

It is remarked that formulas (16), (17), (18) and (19) can be found in [1, p. 257 and p. 259] and [11, 14, 16, 17, 19, 20, 21, 25, 26].

Proof of Theorem 1. Taking logarithm of $g(x)$ leads to

$$\ln g(x) = \frac{\ln \Gamma(x+1) + \ln \Gamma(1/2) - \ln \Gamma(x+1/2) - \ln 2}{x-1}$$

$$\begin{aligned}
&= \frac{\ln \Gamma(x+1) - \ln 2}{x-1} - \frac{\ln \Gamma(x+1/2) - \ln \Gamma(1/2)}{x-1} \\
&= \frac{\ln \Gamma(x+1) - \ln \Gamma(1+1)}{x-1} - \frac{\ln \Gamma(x+1/2) - \ln \Gamma(1+1/2)}{x-1} \\
&= \frac{1}{x-1} \int_1^x \psi(u+1) \, du - \frac{1}{x-1} \int_1^x \psi(u+1/2) \, du \\
&= \frac{1}{x-1} \int_1^x [\psi(u+1) - \psi(u+1/2)] \, du \\
&= \frac{1}{x-1} \int_1^x \int_{1/2}^1 \psi'(u+t) \, dt \, du \\
&= \int_{1/2}^1 \left[\frac{1}{x-1} \int_1^x \psi'(u+t) \, du \right] dt \\
&= \int_{1/2}^1 \int_0^1 \psi'((x-1)u+t+1) \, du \, dt
\end{aligned}$$

and, for $k \in \mathbb{N}$,

$$(-1)^k [\ln g(x)]^{(k)} = \int_{1/2}^1 \int_0^1 u^k [(-1)^k \psi^{(k+1)}((x-1)u+t+1)] \, du \, dt \geq 0$$

by considering formula (18). This means $g(x) \in \mathcal{L}[(-\frac{1}{2}, \infty)]$.

By L'Hospital's rule and formula (17), it is deduced that

$$\begin{aligned}
\lim_{x \rightarrow \infty} \ln g(x) &= \lim_{x \rightarrow \infty} \left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right) \right] \\
&= \lim_{x \rightarrow \infty} \ln \frac{x+1}{x+1/2} + \lim_{x \rightarrow \infty} \int_0^\infty \left(\frac{1}{u} - \frac{1}{1-e^{-u}} \right) (e^{-u} - e^{-u/2}) e^{-xu} \, du = 0,
\end{aligned}$$

which can be restated as $\lim_{x \rightarrow \infty} g(x) = 1$.

It is easy to see that $\lim_{x \rightarrow -\frac{1}{2}+} \ln g(x) = \infty$ and $\lim_{x \rightarrow -\frac{1}{2}+} g(x) = \infty$. The proof of Theorem 1 is complete. \square

Proof of Theorem 2. Using the differences equation $\Gamma(x+1) = x\Gamma(x)$ and taking limit directly gives

$$\lim_{x \rightarrow 0+} h_a(x) = \lim_{x \rightarrow 0} \frac{\Gamma(x+a+1)}{(x+a)^a \Gamma(x+1)} = \frac{\Gamma(1+a)}{a^a}.$$

Using the asymptotic expansion (19) yields

$$h_a(x) = \frac{\Gamma(x+a+1)}{(x+a)^a \Gamma(x+1)} = \left(1 + \frac{a}{x}\right)^{-a} \left[1 + \frac{a(a+1)}{2x} + O\left(\frac{1}{x^2}\right)\right] \rightarrow 1$$

as $x \rightarrow \infty$, which means $\lim_{x \rightarrow \infty} h_a(x) = 1$.

Taking logarithm of $h_a(x)$, differentiating with respect to x successively and utilizing formulas (16) and (18) leads to

$$\ln h_a(x) = (1-a) \ln(x+a) + \ln \Gamma(x+a) - \ln \Gamma(x+1)$$

and, for $n \in \mathbb{N}$ and $n > 1$,

$$[\ln h_a(x)]^{(n)} = (1-a) \frac{(-1)^{n-1} (n-1)!}{(x+a)^n} + \psi^{(n-1)}(x+a) - \psi^{(n-1)}(x+1)$$

$$\begin{aligned}
&= (-1)^n \left[(a-1) \frac{(n-1)!}{(x+a)^n} + (-1)^n \psi^{(n-1)}(x+a) - (-1)^n \psi^{(n-1)}(x+1) \right] \\
&= (-1)^n \left\{ \int_0^\infty (a-1) e^{-(x+a)t} t^{n-1} dt + \int_0^\infty \left[e^{-(x+a)t} - e^{-(x+1)t} \right] \frac{t^{n-1}}{1-e^{-t}} dt \right\} \\
&= (-1)^n \left\{ \int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1-e^{-t}} \left[(a-1)(1-e^{-t}) + 1 - e^{(a-1)t} \right] dt \right\} \\
&\triangleq (-1)^n \int_0^\infty \frac{e^{-(x+a)t} t^{n-1}}{1-e^{-t}} r(t) dt.
\end{aligned}$$

It is clear that $r'(t) = (a-1)e^{-t}(1-e^{at})$ and $r(0) = 0$. Therefore, the function $r(t)$ is non-positive for $a > 1$ and $r(t) \geq 0$ for $0 < a < 1$. As a result, $(-1)^n [\ln h_a(x)]^{(n)} \leq 0$ for $a > 1$ and $(-1)^n [\ln h_a(x)]^{(n)} \geq 0$ for $0 < a < 1$.

By formula (17), it is easy to see that

$$\begin{aligned}
[\ln h_a(x)]' &= \frac{1-a}{x+a} + \psi(x+a) - \psi(x+1) \\
&= \frac{1-a}{x+a} + \ln \frac{x+a}{x+1} + \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}} \right) (e^{-at} - e^{-t}) e^{-xt} dt \\
&\rightarrow 0
\end{aligned}$$

as $x \rightarrow \infty$. Since $[\ln h_a(x)]'$ is decreasing for $a > 1$ and increasing for $0 < a < 1$, then $[\ln h_a(x)]' \geq 0$ for $a > 1$ and $[\ln h_a(x)]' \leq 0$ for $0 < a < 1$.

Summing up, for any positive integer $k \in \mathbb{N}$, if $a > 1$ then $(-1)^k [\ln h_a(x)]^{(k)} \leq 0$, if $0 < a < 1$ then $(-1)^k [\ln h_a(x)]^{(k)} \geq 0$. The proof of Theorem 2 is complete. \square

Proof of Theorem 3. From the differences equation $\Gamma(x+1) = x\Gamma(x)$, it follows easily that

$$\psi(x+1) - \psi(x) = \frac{1}{x} \quad (20)$$

for $x > 0$. Taking logarithm of $p(x)$ and utilizing (20) gives

$$\begin{aligned}
\ln p(x) &= \frac{x \ln x - \ln \Gamma(x+1)}{1-x} \\
&= \frac{\ln \Gamma(x+1) - \ln \Gamma(1+1)}{x-1} - \frac{x \ln x - 1 \ln 1}{x-1} \\
&= \frac{1}{x-1} \int_1^x \psi(u+1) du - \frac{1}{x-1} \int_1^x (1 + \ln u) du \\
&= \frac{1}{x-1} \int_1^x [\psi(u+1) - \ln u] du - 1 \\
&= \frac{1}{x-1} \int_1^x \left[\psi(u) - \ln u + \frac{1}{u} \right] du - 1 \\
&\triangleq \frac{1}{x-1} \int_1^x \Psi(u) du - 1 \\
&= \int_0^1 \Psi((x-1)u+1) du - 1.
\end{aligned}$$

Formulas (16) and (17) imply that

$$\Psi(x) = \psi(x) - \ln x + \frac{1}{x} = \int_0^\infty \left[\frac{e^u - 1 - u}{u(e^u - 1)} \right] e^{-xu} du.$$

Since $e^u - 1 - u > 0$ for $u \in (0, \infty)$, then $\Psi(x) \in \mathcal{C}[(0, \infty)]$. Therefore, for $k \in \mathbb{N}$,

$$(-1)^k [\ln p(x)]^{(k)} = \int_0^1 u^k [(-1)^k \Psi^{(k)}((x-1)u+1)] du \geq 0.$$

This means $p(x) \in \mathcal{L}[(0, \infty)]$.

The L'Hospital's rule and formulas (17) and (20) yield

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln p(x) &= \lim_{x \rightarrow \infty} \frac{x \ln x - \ln \Gamma(x+1)}{1-x} = \lim_{x \rightarrow \infty} [\psi(x+1) - (1 + \ln x)] \\ &= \lim_{x \rightarrow \infty} \left[\psi(x) - \ln x + \frac{1}{x} \right] - 1 = \lim_{x \rightarrow \infty} [\psi(x) - \ln x] + \lim_{x \rightarrow \infty} \frac{1}{x} - 1 = -1. \end{aligned}$$

Thus, it follows easily that $\lim_{x \rightarrow \infty} p(x) = \frac{1}{e}$.

It is clear that $\lim_{x \rightarrow 0+} \ln p(x) = 0$. Hence, the limit $\lim_{x \rightarrow 0+} p(x) = 1$ follows. The proof of Theorem 3 is complete. \square

Proof of Theorem 4. Applying (19) reveals

$$f_a(x) = \frac{\Gamma(x+a)}{x^a \Gamma(x)} = 1 + \frac{a(a-1)}{2x} + O\left(\frac{1}{x^2}\right) \rightarrow 1$$

as $x \rightarrow \infty$ for $a \in (0, \infty)$.

From $f_a(x) = \frac{x^{1-a} \Gamma(x+a)}{\Gamma(x+1)}$, it follows that $\lim_{x \rightarrow 0+} f_a(x) = \begin{cases} \infty, & a > 1, \\ 0, & 0 < a < 1. \end{cases}$

Taking logarithm of $f_a(x)$ and differentiating yields

$$\ln f_a(x) = \ln \Gamma(x+a) - a \ln x - \ln \Gamma(x)$$

and, by (16) and (18) for $n > 1$,

$$\begin{aligned} (-1)^n [\ln f_a(x)]^{(n)} &= (-1)^n \left[\psi^{(n-1)}(x+a) - \psi^{(n-1)}(x) - a \frac{(-1)^{n-1} (n-1)!}{x^n} \right] \\ &= \int_0^\infty (e^{-(x+a)t} - e^{-xt}) \frac{t^{n-1}}{1-e^{-t}} dt + \int_0^\infty a e^{-xt} t^{n-1} dt \\ &= \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} [e^{-at} - 1 + a(1-e^{-t})] e^{-xt} dt \\ &\triangleq \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} s(t) e^{-xt} dt. \end{aligned}$$

It is clear that $s(0) = 0$ and $s'(t) = a[1 - e^{(1-a)t}]e^{-t}$. Thus, standard argument

gives $s(t) \begin{cases} \geq 0, & a > 1, \\ \leq 0, & 0 < a < 1. \end{cases}$ This implies $(-1)^n [\ln f_a(x)]^{(n)} \begin{cases} \geq 0, & a > 1, \\ \leq 0, & 0 < a < 1. \end{cases}$

Since

$$\begin{aligned} [\ln f_a(x)]' &= \psi(x+a) - \psi(x) - \frac{a}{x} \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}} \right) (e^{-at} - 1) e^{-xt} dt + \ln \left(1 + \frac{a}{x} \right) - \frac{a}{x} \rightarrow 0 \end{aligned}$$

as $x \rightarrow \infty$ and the function $[\ln f_a(x)]'$ is increasing for $a > 1$ and decreasing for

$0 < a < 1$, then $[\ln f_a(x)]' \begin{cases} \leq 0, & a > 1, \\ \geq 0, & 0 < a < 1. \end{cases}$

In a word, for $k \in \mathbb{N}$, it follows that $(-1)^k [\ln f_a(x)]^{(k)} \begin{cases} \geq 0, & a > 1, \\ \leq 0, & 0 < a < 1. \end{cases}$ The proof of Theorem 4 is complete. \square

Proof of Theorem 5. As a direct consequence of Theorem 2, a double inequality is obtained: For $0 < a < 1$ and $x > 0$,

$$\left(\frac{x}{x+a}\right)^{1-a} < \frac{\Gamma(x+a)}{x^a \Gamma(x)} < \frac{\Gamma(a+1)}{a^a} \left(\frac{x}{x+a}\right)^{1-a}. \quad (21)$$

For $a > 1$ and $x > 0$, the double inequality (21) is reversed.

As an easy consequence of Theorem 4, an inequality is deduced: For $0 < a < 1$, inequality

$$\frac{\Gamma(x+a)}{x^a \Gamma(x)} < 1 \quad (22)$$

is valid in $x \in (0, \infty)$. For $a > 1$, inequality (22) reverses.

It is a standard argument that $\frac{\Gamma(a+1)}{a^a} \left(\frac{x}{x+a}\right)^{1-a} \leq 1$ if and only if $0 < x \leq \frac{ap(a)}{1-p(a)}$ for $0 < a < 1$. The proof of Theorem 5 is complete. \square

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.
- [2] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.
- [3] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.
- [4] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 100. Available online at <http://jipam.vu.edu.au/article.php?sid=574>.
- [5] M. J. Cloud and B. C. Drachman, *Inequalities with Applications to Engineering*, Springer Verlag, 1998.
- [6] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.
- [7] A. Z. Grinshpan and M. E. H. Ismail, *Completely monotonic functions involving the gamma and q-gamma functions*, Proc. Amer. Math. Soc. **134** (2006), 1153–1160.
- [8] R. A. Horn, *On infinitely divisible matrices, kernels and functions*, Z. Wahrscheinlichkeitstheorie und Verw. Geb **8** (1967), 219–230.
- [9] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.
- [10] L. Lew, J. Frauenthal and N. Keyfitz, *On the average distances in a circular disc*, in: Mathematical Modeling: Classroom Notes in Applied Mathematics, Philadelphia, SIAM, 1987.
- [11] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer, Berlin, 1966.
- [12] F. Qi, *A completely monotonic function involving divided differences of psi and polygamma functions and an application*, submitted.
- [13] F. Qi, *A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum*, submitted.
- [14] F. Qi, *Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at <http://rgmia.vu.edu.au/v8n3.html>.
- [15] F. Qi, *The best bounds in Kershaw's inequality and two completely monotonic functions*, submitted.
- [16] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

- [17] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72. Available online at <http://rgmia.vu.edu.au/v7n1.html>.
- [18] F. Qi and B.-N. Guo, *Some classes of logarithmically completely monotonic functions involving gamma function*, submitted.
- [19] F. Qi, B.-N. Guo, and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436. RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 17. Available online at <http://rgmia.vu.edu.au/v8n2.html>.
- [20] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at <http://rgmia.vu.edu.au/v7n1.html>.
- [21] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **80** (2006), 81–88.
- [22] F. Qi, S.-L. Xu, and L. Debnath, *A new proof of monotonicity for extended mean values*, Internat. J. Math. Math. Sci. **22** (1999), no. 2, 415–420.
- [23] J. Sándor, *On certain inequalities for the Gamma function*, RGMIA Res. Rep. Coll. **9** (2006), no. 1, Art. 11. Available online at <http://rgmia.vu.edu.au/v9n1.html>.
- [24] H. van Haeringen, *Completely Monotonic and Related Functions*, Report 93-108, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands, 1993.
- [25] Zh.-X. Wang and D.-R. Guo, *Special Functions*, Translated from the Chinese by D.-R. Guo and X.-J. Xia, World Scientific Publishing, Singapore, 1989.
- [26] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn*, The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)
- [27] J. Wendel, *Note on the gamma function*, Amer. Math. Monthly **55** (1948), 563–564.
- [28] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

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