PROVING INEQUALITIES IN ACUTE TRIANGLE WITH DIFFERENCE SUBSTITUTION

YU-DONG WU, ZHI-HUA ZHANG, AND YU-RUI ZHANG

ABSTRACT. In this paper, we prove several inequalities in the acute triangle by means of so-called Difference Substitution. As generalization of the method, we also consider an example that the greatest interior angle is less than or equal to 120° in the triangle.

1. INTRODUCTION

In [1, 2], L. Yang suggested make use of a naive method, namely, Difference Substitution, to prove asymmetric polynomial inequalities, as that was used to deal with symmetric ones before.

If $x_1 \le x_2 \le x_3 \le \cdots \le x_n$ with $n \in N^*$, then we set

$$\begin{cases} x_1 = t_1, \\ x_2 = t_1 + t_2, \\ x_3 = t_1 + t_2 + t_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = t_1 + t_2 + t_3 + \cdots + t_n, \end{cases} \tag{1.1}$$

where $t_i \ge 0$ for $2 \le i \le n$ and $i \in N^*$.

The expansion (1.1) is so-called a "splitting" transformation, and $\{t_1, t_2, \cdots, t_n\}$ is just the difference sequence of $\{x_1, x_2, \cdots, x_n\}$.

In general, for the n-variant polynomials, there are n! different orders of $\{x_1, x_2, \cdots, x_n\}$ sorting by size. In the instance of n=3, let $x \le y \le z$, we take

$$\begin{cases} x = u, \\ y = u + v, \\ z = u + v + w, \end{cases} \tag{1.2}$$

where $u \ge 0, v \ge 0, w \ge 0$ if and only if $x \ge y \ge z$.

Analogously, if $y \le x \le z$, then its "splitting" transformation is

$$\begin{cases} y = u, \\ x = u + v, \\ z = u + v + w, \end{cases} \tag{1.3}$$

where $u \ge 0, v \ge 0, w \ge 0$ if and only if $y \ge x \ge z$.

Sequentially, for $y \le z \le x$ or $z \le x \le y$ or $z \le y \le x$ or $x \le z \le y$, we set four similar liner transformations.

For a 3-variant polynomial $F(x, y, z)$, by using six liner transformations above, we obtain 6 members $P_i(u, v, w)$ with $1 \le i \le 6$, and call the set $\{P_1, P_2, \cdots, P_6\}$ the Difference Substitution of $F(x, y, z)$ and denote by $DS(P; F)$. If all the coefficients of these members of $DS(P; F)$ are

Date: July 9, 2005.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Inequalities, acute triangle, difference substitution, liner transformation.

The authors would like to thank Professor Lu Yang for his enthusiastic help.

nonnegative, then $F \geq 0$ whenever x, y, z all are nonnegative, in other words, F is positive semi-definite on R^3_+. Difference substitution is a very valid method to prove inequalities. For more information of Difference Substitution, please refer to [3] and [4].

In this paper, by using Difference Substitution, the authors prove several inequalities in acute triangle.

Throughout the paper we denote A, B, C be the interior angles, a, b, c the side-lengths, S the area, s the semi-perimeter, R the circumradius, r the inradius, h_a, h_b, h_c the altitudes, m_a, m_b, m_c the medians, r_a, r_b, r_c the radii of described circles, and so on. Moreover, we will customarily use the cyclic sum symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, and $\sum f(a, b) = f(a, b) + f(b, c) + f(c, a)$, similarly, one defines others.

Let us begin with the well-known Walker's inequality [5]. In the acute triangle, show that

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{1.4}$$

or

$$\begin{aligned}
&- 2\,a^3b^3 + a^4b^2 - a^4bc + a^5b + ab^5 + b^5c + b^4c^2\\
&- 2\,b^3c^3 + b^2c^4 - 2\,c^3a^3 + c^4a^2 + c^5a + c^5b + c^2a^4\\
&+ a^5c - ab^4c + a^2b^4 - b^6 - c^6 - a^6 - abc^4 \geq 0.
\end{aligned} \tag{1.5}$$

Let

$$\begin{cases} x = \frac{b+c-a}{2} > 0, \\ y = \frac{c+a-b}{2} > 0, \\ z = \frac{a+b-c}{2} > 0. \end{cases} \tag{1.6}$$

Then inequality (1.4) or (1.5) is equivalent to

$$\begin{aligned}
F(x, y, z) = {}& 6\,xyz^4 + 2\,xy^2z^3 + 2\,xy^3z^2 + 6\,xy^4z + 2\,x^2yz^3 + 2\,x^2y^3z\\
&+ 2\,x^3yz^2 + 2\,x^3y^2z + 6\,x^4yz - x^4y^2 - x^2z^4 - 2\,x^3z^3 - x^4z^2\\
&- 2\,x^3y^3 - y^4z^2 - y^4x^2 - 2\,y^3z^3 - y^2z^4 - 18\,x^2y^2z^2 \geq 0.
\end{aligned} \tag{1.7}$$

There is no harm in supposing $x \leq y \leq z$ since inequality (1.7) is symmetric for x, y, z. Then, by using (1.2), for the acute triangle, it follows

$$\begin{aligned}
b^2 + c^2 - a^2 &= (z + x)^2 + (x + y)^2 - (y + z)^2 = 2[x^2 + (y + z)x - yz]\\
&= 2\{u^2 + [(u + v) + (u + v + w)]u - (u + v)(u + v + w)\}\\
&= 2(2u^2 - v^2 - vw) > 0,
\end{aligned} \tag{1.8}$$

and $F(x, y, z)$ in (1.7) is transformed into

$$\begin{aligned}
F(x, y, z) &= P(u, v, w)\\
&= \left(2\,u^2 - v^2 - vw\right) \left[\left(4\,v^2 + 4\,w^2 + 4\,vw\right) u^2\right.\\
&+ \left(8\,v^3 + 20\,vw^2 + 12\,v^2w + 8\,w^3\right) u + 4\,v^4 + 8\,v^3w + 2\,w^4\\
&\left.+ 18\,vw^3 + 22\,v^2w^2\ \right] + \left(24\,v^3w^2 + 36\,v^2w^3 + 12\,vw^4\right) u\\
&+ 34\,v^3w^3 + 19\,v^2w^4 + 2\,vw^5 + 17\,v^4w^2.
\end{aligned} \tag{1.9}$$

This finds obviously $F(x, y, z) = P(u, v, w) \geq 0$ from (1.8) and $u > 0, v \geq 0, w \geq 0$, i.e. inequality (1.4) or (1.5) is true.

Now, let us see another semi-symmetric inequality [6] in the acute triangle

$$\cos(B - C) \leq \frac{h_a}{m_a}. \tag{1.10}$$

It is equivalent to

$$-a^4+\left(3\,b^2+3\,c^2\right)a^2-2\,(b-c)^2\,(b+c)^2\geq 0, \tag{1.11}$$

and from (1.6), this equals

$$F(x,y,z)=\left(-y^2-z^2+14\,yz\right)x^2-(y+z)\left(z^2-14\,yz+y^2\right)x+yz\,(y+z)^2\geq 0. \tag{1.12}$$

Calculating $DB(P;F)$, it consists 3 polynomials with $u>0, v\geq 0, w\geq 0$ as follows

$$\begin{aligned}P_1(u,v,w)=&40\,u^4+112\,u^3v+108\,u^2v^2+56\,u^3w+14\,u^2w^2+40\,uv^3+20\,uvw^2\\&+60\,uv^2w+108\,u^2vw+8\,v^3w+5\,v^2w^2+vw^3+4\,v^4,\end{aligned} \tag{1.13}$$

$$\begin{aligned}P_2(u,v,w)=&\left(2\,u^2-v^2-vw\right)\left(20\,u^2+(24\,w+52\,v)\,u+53\,v^2+6\,w^2+52\,vw\right)\\&+\left(72\,v^3+36\,vw^2+108\,v^2w\right)u+57\,v^2w^2+51\,v^4+6\,vw^3+102\,v^3w,\end{aligned} \tag{1.14}$$

and

$$\begin{aligned}P_3(u,v,w)=&\left(2\,u^2-v^2-vw\right)\left(20\,u^2+(52\,v+28\,w)\,u+53\,v^2+54\,vw+7\,w^2\right)\\&+\left(72\,v^3+36\,vw^2+108\,v^2w\right)u+57\,v^2w^2+51\,v^4+6\,vw^3+102\,v^3w.\end{aligned} \tag{1.15}$$

According to (1.8), we obtain immediately $P_i(u,v,w)\geq 0$ for $1\leq i\leq 3$. Hence, inequality (1.10) is proved.

2. Some Problems and their Proofs

2.1. The Problems.

In 2004-2005, J. Liu [7, 8] posed the following conjectures for the inequality in the acute triangle.

Problem 2.1. *Let $\triangle ABC$ be an acute triangle. Prove the inequalities as follows.*

$$\sum\left(\frac{\sin 2A}{\sin B+\sin C}\right)^2\leq\frac{3}{4}, \tag{2.1}$$

and

$$\sin\frac{A}{2}\leq\frac{\sqrt{m_bm_c}}{2m_a}. \tag{2.2}$$

2.2. The Proof of Inequality 2.1.

Proof. In the facts that $\sin 2\alpha=2\sin\alpha\cos\alpha$, law of cosines and law of sines, we find that inequality (2.1) is equivalent to

$$\begin{aligned}&4\,a^{10}b^2-10\,a^5b^5c^2-24\,b^6c^5a+16\,a^9b^3+4\,a^8b^4-5\,b^4c^6a^2-8\,a^{11}b-5\,a^4b^6c^2+8\,a^9c^2b\\&+4\,a^8b^2c^2-16\,a^{10}cb+8\,a^9cb^2-5\,a^6b^4c^2+32\,a^8b^3c+8\,a^7b^4c+8\,a^7c^4b-5\,a^6c^4b^2\\&+32\,a^8c^3b+4\,a^{10}c^2-8\,a^{11}c-4\,a^{12}-4\,b^{12}-4\,c^{12}+4\,a^8c^4+16\,a^9c^3-8\,a^7b^5-8\,a^6b^6\\&-8\,a^6c^6-8\,a^7c^5-8\,a^5b^7+4\,a^4b^8-24\,a^6b^5c-24\,a^5b^6c+2\,a^5b^3c^4+6\,a^4b^4c^4+4\,c^{10}b^2\\&-26\,a^6b^3c^3+2\,a^5b^4c^3-24\,a^5c^6b-5\,a^4c^6b^2-24\,a^6c^5b-10\,a^5c^5b^2+8\,a^4b^7c-8\,c^{11}b\\&+32\,b^8a^3c+8\,b^9a^2c+4\,b^8a^2c^2+2\,b^5c^3a^4-26\,b^6c^3a^3+2\,b^5c^4a^3-5\,b^6c^4a^2-16\,b^{10}ca\\&+8\,b^9c^2a+32\,b^8c^3a+8\,b^7c^4a-10\,b^5c^5a^2+16\,b^9a^3+4\,b^{10}a^2-8\,b^{11}a-8\,b^{11}c\\&+4\,b^{10}c^2+16\,b^9c^3+4\,b^8c^4-8\,b^7c^5-8\,b^6c^6+4\,a^4c^8-8\,a^5c^7-24\,b^5c^6a+8\,a^4c^7b\\&+2\,c^5b^4a^3-26\,c^6b^3a^3+2\,c^5b^3a^4+4\,c^8b^2a^2+8\,c^9a^2b+32\,c^8a^3b+8\,b^4c^7a-8\,c^{11}a\\&+32\,c^8b^3a+8\,c^9b^2a-16\,c^{10}ab+4\,c^{10}a^2+16\,c^9a^3-8\,b^5c^7+4\,b^4c^8+16\,c^9b^3\geq 0.\end{aligned} \tag{2.3}$$

From (1.6), then inequality (2.3) equals

$$\begin{aligned}
F(x,y,z) =& - 4576\,x^7z^5 - 5590\,x^6z^6 - 116\,x^{10}z^2 - 2453\,x^4z^8 - 2453\,x^8z^4 - 4576\,x^5z^7 - 788\,x^3z^9 \\
& - 788\,x^9z^3 - 2453\,x^8y^4 - 4576\,y^7z^5 - 2453\,y^8z^4 - 2453\,x^4y^8 - 788\,x^9y^3 - 788\,x^3y^9 \\
& - 5590\,x^6y^6 - 4576\,x^7y^5 - 4576\,x^5y^7 - 2453\,y^4z^8 - 4576\,y^5z^7 - 5590\,y^6z^6 - 116\,y^{10}z^2 \\
& - 788\,y^9z^3 - 116\,y^{10}x^2 - 788\,y^3z^9 - 116\,y^2z^{10} + 13448\,x^6y^5z + 8176\,x^7yz^4 + 13448\,x^6yz^5 \\
& + 13448\,x^5yz^6 + 8176\,x^4yz^7 + 6448\,x^2y^3z^7 + 1220\,xy^9z^2 + 6448\,x^2y^7z^3 + 6448\,x^3y^7z^2 \\
& + 1220\,x^9yz^2 + 10862\,x^4y^6z^2 + 14288\,x^2y^5z^5 + 10862\,x^2y^4z^6 + 14288\,x^5y^2z^5 \\
& + 10862\,x^6y^2z^4 + 6448\,x^7y^2z^3 - 28248\,x^3y^5z^4 - 28248\,x^4y^5z^3 + 14288\,x^5y^5z^2 \\
& - 57474\,x^4y^4z^4 - 28248\,x^3y^4z^5 - 8672\,x^3y^3z^6 + 6448\,x^3y^2z^7 + 10862\,x^2y^6z^4 \\
& - 8672\,x^3y^6z^3 - 28248\,x^5y^4z^3 + 10862\,x^6y^4z^2 - 28248\,x^4y^3z^5 - 28248\,x^5y^3z^4 \\
& - 8672\,x^6y^3z^3 + 6448\,x^7y^3z^2 + 10862\,x^4y^2z^6 + 3420\,x^8yz^3 + 3420\,x^8y^3z \\
& + 1220\,x^2yz^9 + 280\,xy^{10}z + 4066\,x^2y^2z^8 + 3420\,x^3yz^8 + 3420\,x^3y^8z + 4066\,x^8y^2z^2 \\
& + 4066\,x^2y^8z^2 + 1220\,xy^2z^9 + 8176\,x^7y^4z + 3420\,xy^3z^8 + 3420\,xy^8z^3 + 1220\,x^2y^9z \\
& + 280\,xyz^{10} + 280\,x^{10}yz + 1220\,x^9y^2z + 8176\,xy^4z^7 + 13448\,xy^5z^6 + 13448\,xy^6z^5 \\
& + 8176\,x^4y^7z + 8176\,xy^7z^4 + 13448\,x^5y^6z - 116\,x^{10}y^2 - 116\,x^2z^{10} \geq 0.
\end{aligned} \tag{2.4}$$

Since (2.4) is symmetric inequality for x, y, z, there is no harm in supposing $x \leq y \leq z$. Using transformation (1.2), then $F(x,y,z)$ in (2.4) is became into

$$\begin{aligned}
F(x,y,z) =& P(u,v,w) \\
=& (2u^2 - v^2 - vw)[(180224\,w^2 + 180224\,v^2 + 180224\,vw)u^8 + (1794048\,v^2w \\
& + 1810432\,vw^2 + 606208\,w^3 + 1196032\,v^3)u^7 + (4360192\,vw^3 + 7030784\,v^3w \\
& + 771072\,w^4 + 7875584\,v^2w^2 + 3515392\,v^4)u^6 + (6049280\,v^5 + 520704\,w^5 \\
& + 19394048\,v^3w^2 + 13967872\,v^2w^3 + 4689152\,vw^4 + 15123200\,v^4w)u^5 \\
& + (2838144\,vw^5 + 6838400\,v^6 + 12647648\,v^2w^4 + 30324704\,v^4w^2 + 210048\,w^6 \\
& + 26457408\,v^3w^3 + 20515200\,v^5w)u^4 + (19291776\,v^6w + 52480\,w^7 + 1074176\,vw^6 \\
& + 5511936\,v^7 + 32787968\,v^5w^2 + 33740480\,v^4w^3 + 20662912\,v^3w^4 + 6899776\,v^2w^5)u^3 \\
& + (32727200\,v^5w^3 + 7968\,w^8 + 2528912\,w^6v^2 + 24395856\,v^4w^4 + 27385760\,v^6w^2 \\
& + 14122880\,v^7w + 268096\,w^7v + 3530720\,v^8 + 10723072\,v^3w^5)u^2 + (9558576\,v^8w \\
& + 4185944\,v^3w^6 + 13383144\,v^4w^5 + 676240\,v^2w^7 + 2124128\,v^9 + 672\,w^9 \\
& + 24737624\,v^5w^4 + 45200\,vw^8 + 20832112\,v^7w^2 + 28305704\,v^6w^3)u + 15686836\,v^5w^5 \\
& + 6092840\,v^4w^6 + 1326664\,v^3w^7 + 139150\,v^2w^8 + 24651416\,v^6w^4 + 24921352\,v^7w^3 \\
& + 1378920\,v^{10} + 6894600\,v^9w + 16572238\,v^8w^2 + 5112\,vw^9 + 24\,w^{10}] + (27659640\,v^9w^2 \\
& + 10558592\,v^{10}w + 689380\,v^3w^8 + 4642800\,v^4w^7 + 36001700\,v^6w^5 + 45278940\,v^8w^3 \\
& + 16715660\,v^5w^6 + 720\,vw^{10} + 49540936\,v^7w^4 + 1919744\,v^{11} + 44048\,v^2w^9)u \\
& + 5020\,v^2w^{10} + 49008067\,v^8w^4 + 142314\,v^3w^9 + 23121662\,v^{10}w^2 + 8144784\,v^{11}w \\
& + 1451049\,v^4w^8 + 40947790\,v^9w^3 + 24\,vw^{11} + 7353016\,v^5w^7 + 1357464\,v^{12} \\
& + 39938152\,v^7w^5 + 21582818\,v^6w^6.
\end{aligned} \tag{2.5}$$

This implies $F(x,y,z) = P(u,v,w) \geq 0$ from (1.8) in the acute triangle. Hence, inequality (2.1) holds. The proof is completed. ∎

2.3. **The Proof of Inequality** (2.2).

Proof. Inequality (2.2) is equivalent to

$$\sin^4 \frac{A}{2} \leq \frac{m_b^2 m_c^2}{16 m_a^4}. \tag{2.6}$$

By using the formula $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, law of cosines and the formulas of the medians, we find that (2.6) is just the following inequality

$$\begin{aligned} &- a^8 - 4\,b^8 + 6\,a^6c^2 - 34\,b^2c^6 + 20\,b^3a^4c + 12\,a^2c^6 - 32\,b^5a^2c - 32\,c^5a^2b \\ &- 34\,b^6c^2 - 51\,b^4c^4 - 4\,c^8 - 4\,a^6bc + 20\,c^3a^4b - 26\,a^4b^2c^2 + 54\,a^2b^4c^2 \\ &+ 54\,a^2b^2c^4 - 13\,a^4b^4 - 13\,a^4c^4 + 12\,a^2b^6 + 6\,a^6b^2 + 16\,b^7c + 16\,c^7b \\ &- 64\,b^3c^3a^2 + 48\,b^5c^3 + 48\,b^3c^5 \geq 0. \end{aligned} \tag{2.7}$$

Considering (1.6), then inequality (2.7) is transformed into

$$\begin{aligned} F(x,y,z) = {} & x^8 + (4\,z + 4\,y)x^7 + (2\,z^2 + 40\,yz + 2\,y^2)x^6 + (-8\,z^3 + 84\,yz^2 - 8\,y^3 + 84\,y^2z)x^5 \\ & + (-20\,y^2z^2 + 76\,yz^3 + 76\,y^3z - 7\,z^4 - 7\,y^4)x^4 + (4\,z^5 + 48\,y^4z - 248\,y^3z^2 - 248\,y^2z^3 \\ & + 4\,y^5 + 48\,yz^4)x^3 + (4\,y^6 - 234\,y^4z^2 - 234\,y^2z^4 + 28\,y^5z + 4\,z^6 + 28\,yz^5 + 32\,y^3z^3)x^2 \\ & + (-84\,z^5y^2 + 8\,z^6y - 84\,y^5z^2 + 256\,y^3z^4 + 256\,y^4z^3 + 8\,y^6z)x + 84\,z^5y^3 - 63\,z^4y^4 \\ & - 12\,y^6z^2 - 12\,z^6y^2 + 84\,y^5z^3 \geq 0. \end{aligned} \tag{2.8}$$

It's easy to find that inequality (2.8) is symmetric for y, z. Therefor, we only need to prove that inequality (2.8) holds when $x \leq y \leq z$, $y \leq x \leq z$ and $y \leq z \leq x$.

Calculating $DB(P;F)$, it consists 3 polynomials with $u > 0, v \geq 0, w \geq 0$ as follows

$$\begin{aligned} P_1(u,v,w) = {} & (2u^2 - v^2 - vw)[(192\,w^2 + 768\,v^2 + 768\,vw)u^4 + (256\,w^3 + 2112\,vw^2 \\ & + 4800\,v^2w + 3200\,v^3)u^3 + (5808\,v^4 + 80\,w^4 + 7376\,v^2w^2 + 11616\,v^3w + 1568\,vw^3)u^2 \\ & + (6336\,v^5 + 15840\,v^4w + 13440\,v^3w^2 + 16\,w^5 + 4320\,v^2w^3 + 416\,vw^4)u + 5112\,v^6 \\ & + 15336\,v^5w + 48\,vw^5 + 16560\,v^4w^2 + 7560\,v^3w^3 + 1272\,v^2w^4] + (7344\,v^7 + 432\,w^5v^2 \\ & + 25704\,v^6w + 33912\,v^5w^2 + 20520\,v^4w^3 + 5400\,v^3w^4)u + 20772\,v^7w + 5193\,v^8 \\ & + 36\,w^6v^2 + 1332\,w^5v^3 + 32418\,v^6w^2 + 24552\,v^5w^3 + 9009\,v^4w^4 \end{aligned} \tag{2.9}$$

for $x \leq y \leq z$,

$$\begin{aligned} P_2(u,v,w) = {} & (2u^2 - v^2 - vw)[(-384\,vw + 192\,v^2 + 192\,w^2)u^4 + (-192\,vw^2 + 896\,v^3 \\ & - 960\,v^2w + 256\,w^3)u^3 + (-976\,v^2w^2 + 1776\,v^4 + 80\,w^4 + 224\,vw^3 - 288\,v^3w)u^2 \\ & + (2032\,v^5 - 480\,v^3w^2 + 16\,w^5 + 1328\,v^4w + 128\,v^2w^3 + 240\,vw^4)u + 1640\,v^6 \\ & + 2128\,v^5w + 544\,v^4w^2 + 328\,v^3w^3 + 416\,v^2w^4 + 80\,vw^5] + (2064\,v^5w^2 + 32\,w^6v \\ & + 4176\,v^6w + 776\,v^4w^3 + 2320\,v^7 + 416\,v^2w^5 + 968\,v^3w^4)u + 1640\,v^8 + 2708\,w^2v^6 \\ & + 817\,w^4v^4 + 524\,w^5v^3 + 956\,w^3v^5 + 84\,w^6v^2 + 3768\,wv^7 \end{aligned} \tag{2.10}$$

for $y \le x \le z$, and

$$\begin{aligned}P_3(u,v,w) =&\ 384\,u^6v^2 + 11072\,w^2u^2v^4 + 20992\,w^2u^3v^3 + 19552\,w^2u^4v^2 + 8832\,w^2u^5v\\&+ 2008\,w^4uv^3 + 5296\,w^4u^2v^2 + 5376\,w^4u^3v + 36\,w^2v^6 + 1536\,w^2u^6 + 2792\,w^3uv^4\\&+ 10400\,w^3u^2v^3 + 15744\,w^3u^3v^2 + 10816\,w^3u^4v + 2368\,w^2uv^5 + 840\,w^5uv^2\\&+ 1344\,w^5u^2v + 2816\,w^3u^5 + 132\,w^3v^5 + 1888\,w^4u^4 + 193\,w^4v^4 + 184\,w^6uv + 144\,w^5v^3\\&+ 640\,w^5u^3 + 1200\,wuv^6 + 13120\,wu^3v^4 + 6256\,wu^2v^5 + 13824\,wu^4v^3 + 58\,w^6v^2\\&+ 128\,w^6u^2 + 7296\,wu^5v^2 + 3360\,u^4v^4 + 1792\,u^5v^3 + 288\,uv^7 + 1504\,u^2v^6 + 3168\,u^3v^5\\&+ 1536\,wu^6v + 12\,w^7v + w^8 + 16\,w^7u\end{aligned} \tag{2.11}$$

for $y \le z \le x$.

There are not difficult to see that $P_1(u,v,w) \ge 0$ and $P_3(u,v,w) \ge 0$ because $u > 0, v \ge 0, w \ge 0$ and $2u^2 - v^2 - vw > 0$.

In order to prove $P_2(u,v,w) \ge 0$, we only need prove the following inequality also.

$$\begin{aligned}p(u,v,w) =&(-384\,vw + 192\,v^2 + 192\,w^2)u^4 + (-192\,vw^2 + 896\,v^3 - 960\,v^2w + 256\,w^3)u^3\\&+ (-976\,v^2w^2 + 1776\,v^4 + 80\,w^4 + 224\,vw^3 - 288\,v^3w)u^2 + (2032\,v^5 - 480\,v^3w^2\\&+ 16\,w^5 + 1328\,v^4w + 128\,v^2w^3 + 240\,vw^4)u + 1640\,v^6 + 2128\,v^5w + 544\,v^4w^2\\&+ 328\,v^3w^3 + 416\,v^2w^4 + 80\,vw^5 \ge 0,\end{aligned} \tag{2.12}$$

where $u > 0$, $v \ge 0$ and $w \ge 0$.

(i) For $u > 0, v \ge w \ge 0$, taking $v = w + t$ with $t \ge 0$, then we have

$$\begin{aligned}p(u,v,w) =&192\,t^2u^4 + (576\,tw^2 + 1728\,t^2w + 896\,t^3)u^3 + (816\,w^4 + 4512\,w^3t + 8816\,w^2t^2\\&+ 6816\,wt^3 + 1776\,t^4)u^2 + (2032\,t^5 + 11488\,wt^4 + 3264\,w^5 + 14528\,w^4t\\&+ 26976\,w^3t^2 + 25152\,w^2t^3)u + 50544\,w^4t^2 + 56584\,w^3t^3 + 5136\,w^6\\&+ 24552\,w^5t + 1640\,t^6 + 35784\,w^2t^4 + 11968\,wt^5.\end{aligned}$$

It is obviously to follow $p(u,v,w) \ge 0$, i.e. inequality (2.12) holds.

(ii) When $u > 0, w \ge v \ge 0$, setting $w = v + t$ for $t \ge 0$, then we get

$$\begin{aligned}p(u,v,w) =&(2\,u + 10\,v)t^5 + (10\,u^2 + 40\,uv + 102\,v^2)t^4 + (156\,uv^2 + 32\,u^3 + 349\,v^3 + 68\,u^2v)t^3\\&+ (24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4)t^2 + v^2(783\,v^3 - 72\,u^3 + 224\,uv^2\\&- 156\,u^2v)t + 6\,v^4(17\,u^2 + 68\,uv + 107\,v^2) = p_1(u,v,t) + p_2(u,v,t),\end{aligned}$$

where

$$p_1(u,v,t) = (2\,u + 10\,v)t^5 + (10\,u^2 + 40\,uv + 102\,v^2)t^4 + (156\,uv^2 + 32\,u^3 + 349\,v^3 + 68\,u^2v)t^3 \ge 0,$$

and

$$\begin{aligned}p_2(u,v,t) =&(24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4)t^2\\&+ v^2(783\,v^3 - 72\,u^3 + 224\,uv^2 - 156\,u^2v)t + 6\,v^4(17\,u^2 + 68\,uv + 107\,v^2).\end{aligned} \tag{2.13}$$

It's easily to find that $24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4 > 0$, and the discriminant of quadratic function (2.13) with respect to t is

$$\begin{aligned}\Delta(u,v) =& - v^4(935415\,v^6 + 480144\,u^3v^3 + 1116096\,uv^5\\&+ 803456\,u^2v^4 + 4608\,u^6 + 196032\,u^4v^2 + 46080\,u^5v) \le 0.\end{aligned}$$

This is to say that $p_2(u,v,t) \ge 0$.

Hence, $P_2(u,v,w) \ge 0$. From the proof above, the required result (2.6) is proved. ∎

2.4. **Remarks.**

Remark 2.1. By the same argument as above, we also prove the following inequalities conjectures [9, 10, 11] in the acute triangle.

$$\sum m_a^2 h_a^2 \geq \sum m_a^2 r_a^2, \tag{2.14}$$

$$\sum sin^8 A \geq \sum cos^8 \frac{A}{2}, \tag{2.15}$$

$$\sum (b-c)^2 \geq \sum \left(\frac{a}{b+c}\right)^2 (r_b - r_c)^2, \tag{2.16}$$

and

$$\sum (h_b + h_c - h_a)^3 \geq 3 m_a m_b m_c. \tag{2.17}$$

Remark 2.2. We do the operation in this paper with mathematical software Maple 9.0.

3. Generalization of the Method

In fact, Difference Substitution can go more far. Now, we consider the following inequality [12]. In $\triangle ABC$, if $\max(A, B, C) \leq \frac{2\pi}{3}$, then

$$s^2 \geq R^2 + 10Rr + 3r^2. \tag{3.1}$$

Utilizing the known formulas $R = \frac{abc}{4S}$, $r = \frac{S}{s}$ and $S = \sqrt{s(s-a)(s-b)(s-c)}$, and from (1.6), inequality (3.1) is equivalent to

$$\begin{aligned} &3\,c^2a^2b^2 - a^2bc^3 - a^3bc^2 - a^3b^2c - a^2b^3c - ab^2c^3 - ab^3c^2 - 2\,b^3c^3 \\ &- 2\,a^3c^3 + c^4b^2 - 2\,a^3b^3 + c^5b + a^4c^2 + b^4c^2 + b^5a + a^5c + c^5a \\ &- a^6 + a^4b^2 + a^5b - b^6 - c^6 + a^2b^4 + b^5c + a^2c^4 \geq 0, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} F(x,y,z) = &- 42\,x^2y^2z^2 + 14\,y^4zx + 14\,xyz^4 + 2\,xy^2z^3 + 2\,x^2y^3z + 2\,xy^3z^2 + 14\,x^4yz \\ &+ 2\,x^3yz^2 + 2\,x^2yz^3 + 2\,x^3y^2z - x^4y^2 - x^2z^4 - 2\,x^3z^3 - x^4z^2 - 2\,x^3y^3 \\ &- y^4z^2 - y^4x^2 - 2\,y^3z^3 - y^2z^4 \geq 0, \end{aligned} \tag{3.3}$$

where $x > 0, y > 0, z > 0$.

For inequality (3.3) is symmetric with x, y, z, there is no harm in supposing $x \leq y \leq z$. From (1.2), then $F(x, y, z)$ in (3.3) is transformed into

$$\begin{aligned} F(x,y,z) &= P(u,v,w) \\ &= (8\,u^2 + 4\,uv + 2\,uw - v^2 - vw)(8\,uvw^2 + 12\,uv^2w + 4\,u^2vw + 4\,v^4 \\ &+ 4\,u^2v^2 + 2\,uw^3 + 8\,uv^3 + 8\,v^3w + 7\,v^2w^2 + 3\,vw^3) \\ &+ 2\,w^2(v + 2\,u)^2(v + 2\,u + w)^2 \geq 0, \end{aligned} \tag{3.4}$$

and for $\max(A, B, C) \leq \frac{2\pi}{3}$ and the function $y = \cos x$ decreases in $x \in (0, \pi)$, yields

$$\begin{aligned} b^2 + c^2 + bc - a^2 &= b^2 + c^2 - \frac{1}{2}bc\cos\frac{2\pi}{3} - a^2 \\ &= 3x^2 + 3(y+z)x - yz \\ &= 8u^2 + 4uv + 2uw - v^2 - vw \\ &\geq b^2 + c^2 - \frac{1}{2}bc\cos A - a^2 = 0. \end{aligned} \tag{3.5}$$

These are deduced obviously $F(x, y, z) = P(u, v, w) \geq 0$ for $u > 0, v \geq 0$ and $w \geq 0$, i.e. inequality (3.1) is obtained.

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(Y.-D. Wu) Xinchang High School, Xinchang City, Zhejiang Province 312500, P. R. China.
E-mail address: zjxcwyd@tom.com

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou City, Hunan Province 423400, P. R. China
E-mail address: zxzh1234@163.com
URL: http://www.zxslzx.com/zzhweb/

(Y.-R. Zhang) Xinchang High School, Xinchang City, Zhejiang Province 312500, P. R. China.
E-mail address: xczxzyr@163.com