SOME APPLICATIONS OF DE BRUIJN'S INEQUALITY FOR POWER SERIES

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Abstract. Some inequalities for power series with real coefficients via the de Bruijn inequality are established. Applications for fundamental complex functions such as the exp, cos, sin, cosh, sinh and ln are also given.

1. Introduction

If we consider an analytic function $f(z)$ defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and apply the well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2, \tag{1.1}$$

holding for the complex numbers a_j, b_j, $j \in \{1, \dots, n\}$, then we can deduce that

$$|f(z)|^2 = \left|\sum_{n=0}^{\infty} a_n z^n\right|^2 \le \sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1-|z|^2} \cdot \sum_{n=0}^{\infty} |a_n|^2 \tag{1.2}$$

for any $z \in D(0, R) \cap D(0, 1)$, where R is the radius of convergence of f.

The above inequality gives some information about the magnitude of the function f provided that numerical series $\sum_{n=0}^{\infty} |a_n|^2$ is convergent and z is not too close to the boundary of the open disk $D(0, 1)$.

It is then natural to ask the question as to whether we can obtain better bounds for the magnitude of f if the coefficients are restricted to be real numbers, a case which is of large interest due to the fact that many usual complex functions can be represented as power series with real coefficients.

If we restrict ourselves more and assume that the coefficients in the representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are nonnegative, and the assumption incorporates various examples of complex functions that will be indicated in the sequel, on utilising the weighted version of the CBS-inequality, namely

$$\left|\sum_{j=1}^{n} w_j a_j b_j\right|^2 \le \sum_{j=1}^{n} w_j |a_j|^2 \sum_{j=1}^{n} w_j |b_j|^2, \tag{1.3}$$

Date: 16 October, 2006.
2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. De Bruijn inequality, Power series, Complex functions.

where $w_j \geq 0$, while $a_j, b_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$, we can state that

$$(1.4) \quad |f(zw)|^2 = \left| \sum_{n=0}^{\infty} a_n z^n w^n \right|^2 \leq \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} = f\left(|z|^2\right) f\left(|w|^2\right)$$

for any $z, w \in \mathbb{C}$ with $zw, |z|^2, |w|^2 \in D(0, R)$.

The problem of improving (1.4) for one of the numbers z or w assumed to be real is then also natural to be considered and the obtained result would give interesting examples of inequalities for real and complex numbers.

Motivated by the above questions and utilising a tool that has been available in the literature since 1960 and known as the *de Bruijn inequality*, we establish in this paper some inequalities for functions defined by power series which will improve the above results. Particular examples that are related to some fundamental complex functions such as the exp, cos, sin, cosh, sinh and ln are presented.

2. General Results via the de Bruijn Inequality

In an effort to provide a refinement for the celebrated Cauchy-Bunyakovsky-Schwarz inequality for complex numbers

$$(2.1) \quad \left| \sum_{j=1}^{n} a_j b_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2,$$

de Bruijn established in 1960, [1] (see also [3, p. 89] or [2, p. 48]) the following result:

Lemma 1 (de Bruijn, 1960)**.** *If* $\mathbf{b} = (b_1, \dots, b_n)$ *is an* $n-$*tuple of real numbers and* $z = (z_1, \dots, z_n)$ *an* $n-$*tuple of complex numbers, then*

$$(2.2) \quad \left| \sum_{k=1}^{n} b_k z_k \right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} b_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left| \sum_{k=1}^{n} z_k^2 \right| \right].$$

Equality holds in (2.2) if and only if for $k \in \{1, \dots, n\}$, $b_k = \operatorname{Re}(\lambda z_k)$, *where* λ *is a complex number such that the quantity* $\lambda^2 \sum_{k=1}^{n} z_k^2$ *is a nonnegative real number.*

On utilising this result, we can establish some inequalities for power series as follows:

Theorem 1. *Let* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *be an analytic function defined by a power series with nonnegative coefficients* a_n, $n \in \mathbb{N}$ *and convergent on the open disk* $D(0, R) \subset \mathbb{C}$, $R > 0$. *If* a *is a real number and* z *a complex number such that* az, $a^2, z^2, |z|^2 \in D(0, R)$, *then:*

$$(2.3) \quad |f(az)|^2 \leq \frac{1}{2} f\left(a^2\right) \left[f\left(|z|^2\right) + \left| f\left(z^2\right) \right| \right].$$

Proof. First of all, notice that, by de Bruijn's inequality (2.2) for the choices $b_k = \sqrt{a_k} c_k$, $z_k = \sqrt{a_k} w_k$ with $a_k \geq 0$, $c_k \in \mathbb{R}$ and $w_k \in \mathbb{C}$, $k \in \{0, \dots, n\}$, we can state the weighted inequality:

$$(2.4) \quad \left| \sum_{k=0}^{n} a_k c_k w_k \right|^2 \leq \frac{1}{2} \sum_{k=0}^{n} a_k c_k^2 \left[\sum_{k=0}^{n} a_k |w_k|^2 + \left| \sum_{k=0}^{n} a_k w_k^2 \right| \right].$$

Now, on making use of (2.4) for the partial sums of the function f defined above, we are able to state that:

$$\left|\sum_{k=0}^{n} a_k a^k z^k\right|^2 \leq \frac{1}{2}\sum_{k=0}^{n} a_k a^{2k}\left[\sum_{k=0}^{n} a_k |z|^{2k} + \left|\sum_{k=0}^{n} a_k z^{2k}\right|\right] \tag{2.5}$$

for any $n \geq 0$, $a \in \mathbb{R}$, $z \in \mathbb{C}$ with $az, a^2, |z|^2, z^2 \in D(0,R)$.

Taking the limit as $n \to \infty$ in (2.5) and noticing that all the involved series are convergent, we deduce the desired inequality (2.3). ∎

The inequality (2.3) is a valuable source of particular inequalities for real numbers a and complex numbers z as will be outlined in the following.

(1) If in (2.3) we choose the function $f(z) = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we can state that

$$\left|\frac{1}{1-az}\right|^2 \leq \frac{1}{2}\cdot\frac{1}{1-a^2}\left[\frac{1}{1-|z|^2} + \frac{1}{|1-z^2|}\right] \tag{2.6}$$

for any $a \in (-1,1)$ and $z \in D(0,1)$. In an equivalent form, (2.6) can be stated as

$$\left[1-|z|^2 + \left|1-z^2\right|\right]|1-az|^2 \geq 2\left(1-a^2\right)\left(1-|z|^2\right)\left|1-z^2\right| \tag{2.7}$$

for any $a \in (-1,1)$ and $z \in D(0,1)$.

A direct proof of (2.7) is, in our opinion, difficult and we leave as an open problem to the reader to find alternative proofs of (2.7) without the use of the de Bruijn inequality which has been used as a key point in obtaining the result (2.3).

(2) The choice of another fundamental power series, $f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, will produce via the inequality (2.3) the following result:

$$|\exp(az)|^2 \leq \frac{1}{2}\exp\left(a^2\right)\left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right|\right] \tag{2.8}$$

for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

The choice $z = i$ in (2.8) generates the following simple and interesting result, for which, the authors were not able to find any reference in the literature:

$$|\exp(ia)|^2 \leq \frac{e^2+1}{2e}\exp\left(a^2\right), \tag{2.9}$$

for any $a \in \mathbb{R}$.

(3) Now, if we choose the power series $f(z) = \ln(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n$, $z \in D(0,1)$ and apply Theorem 1, then we get the following inequality for logarithms:

$$|\ln(1-az)|^2 \leq \frac{1}{2}\ln\left(\frac{1}{1-a^2}\right)\left[\ln\left(\frac{1}{1-|z|^2}\right) + \left|\ln\left(1-z^2\right)\right|\right] \tag{2.10}$$

for any $a \in (-1,1)$ and $z \in D(0,1)$.

If in (2.10) we choose $z = \pm ib$ with $b \in (-1,1)$, then we obtain the simpler result:

$$|\ln(1 \pm iab)|^2 \leq \frac{1}{2}\ln\left(\frac{1}{1-a^2}\right)\ln\left(\frac{1+b^2}{1-b^2}\right) \tag{2.11}$$

for any $a, b \in (-1, 1)$.

(4) If we utilise the following function as power series representations with nonnegative coefficients:

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \qquad z \in \mathbb{C};$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \qquad z \in \mathbb{C};$$

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\,n!} z^{2n+1}, \qquad z \in D(0,1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1)$$

$${}_2F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \quad \alpha, \beta, \gamma > 0, z \in D(0,1);$$

where Γ is the *Gamma function*, then we can state the following inequalities:

$$|\cosh(az)|^2 \le \frac{1}{2}\cosh\left(a^2\right)\left[\cosh\left(|z|^2\right) + \left|\cosh\left(z^2\right)\right|\right], \quad a \in \mathbb{R},\ z \in \mathbb{C}; \tag{2.12}$$

$$|\sinh(az)|^2 \le \frac{1}{2}\sinh\left(a^2\right)\left[\sinh\left(|z|^2\right) + \left|\sinh\left(z^2\right)\right|\right], \quad a \in \mathbb{R},\ z \in \mathbb{C}; \tag{2.13}$$

$$\left|\ln\left(\frac{1+az}{1-az}\right)\right| \le \frac{1}{2}\ln\left(\frac{1+a^2}{1-a^2}\right)\left[\ln\left(\frac{1+|z|^2}{1-|z|^2}\right) + \left|\ln\left(\frac{1+z^2}{1-z^2}\right)\right|\right] \tag{2.14}$$
$$a \in (-1,1), \quad z \in D(0,1);$$

$$\left|\sin^{-1}(az)\right|^2 \le \frac{1}{2}\sin^{-1}\left(a^2\right)\left[\sin^{-1}\left(|z|^2\right) + \left|\sin^{-1}\left(z^2\right)\right|\right], \tag{2.15}$$
$$a \in (-1,1), \quad z \in D(0,1);$$

$$\left|\tanh^{-1}(az)\right|^2 \le \frac{1}{2}\tanh^{-1}\left(a^2\right)\left[\tanh^{-1}\left(|z|^2\right) + \left|\tanh^{-1}\left(z^2\right)\right|\right], \tag{2.16}$$
$$a \in (-1,1), \quad z \in D(0,1);$$

and

$$\left|{}_2F_1(\alpha, \beta, \gamma, az)\right|^2 \le \frac{1}{2} \cdot {}_2F_1\left(\alpha, \beta, \gamma, a^2\right)\left[{}_2F_1\left(\alpha, \beta, \gamma, |z|^2\right) + {}_2F_1\left(\alpha, \beta, \gamma, z^2\right)\right] \tag{2.17}$$
$$a \in (-1,1), \quad z \in D(0,1), \quad \alpha, \beta, \gamma > 0.$$

Now, by the help of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series.

As some natural examples that are useful for applications, we can point out that, if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n=\ln\frac{1}{1+z},\qquad z\in D(0,1);$$
$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\qquad z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\qquad z\in\mathbb{C};$$
$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\qquad z\in D(0,1),$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are obviously

$$f_A(z)=\ln\frac{1}{1-z},\qquad g_A(z)=\cosh z,$$
$$h_A(z)=\sinh z\quad\text{and}\quad l_A(z)=\frac{1}{1-z}$$

and they are defined on the same domain as the generating functions.

The following result contains an inequality connecting the function f with its transform f_A.

Theorem 2. *Let $f(z):=\sum_{n=0}^{\infty}a_nz^n$ be a function defined by a power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If $a\in\mathbb{R}$ an $z\in\mathbb{C}$ are such that $az,a^2,|z|^2,z^2\in D(0,R)$, then:*

$$|f(az)|^2\le\frac{1}{2}f_A\left(a^2\right)\left[f_A\left(|z|^2\right)+\left|f_A\left(z^2\right)\right|\right].\tag{2.18}$$

Proof. Firstly, observe that for each $n\in\mathbb{N}$, $a_n=|a_n|\operatorname{sgn}(a_n)$ where $\operatorname{sgn}(x)$ is the sign function defined to be 1 if $x>0$, -1 if $x<0$ and 0 if $x=0$.

Utilising the de Bruijn inequality with positive weights we have

$$\begin{aligned}|f(az)|^2&=\left|\sum_{n=0}^{\infty}|a_n|\operatorname{sgn}(a_n)a^nz^n\right|^2\\&\le\frac{1}{2}\sum_{n=0}^{\infty}|a_n|\left[\operatorname{sgn}(a_n)\right]^2a^{2n}\left[\sum_{n=0}^{\infty}|a_n||z|^{2n}+\sum_{n=0}^{\infty}|a_n|\left|z^{2n}\right|\right]\\&=\frac{1}{2}f_A\left(a^2\right)\left[f_A\left(|z|^2\right)+\left|f_A\left(z^2\right)\right|\right]\end{aligned}$$

for any $a\in\mathbb{R}$, $z\in\mathbb{C}$ with $az,a^2,|z|^2,z^2\in D(0,R)$. ∎

In the following examples we exemplify how the above inequality may be used to establish some inequalities for real and complex numbers that otherwise would be very difficult to prove.

(1) If we take the function

$$f(z) = \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n+1}, \qquad z \in \mathbb{C},$$

then

$$f_A(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right) \quad \text{for } z \in \mathbb{C}.$$

Applying the inequality (2.18) for this selection will produce the result

$$|\sin(az)|^2 \le \frac{1}{2} \sinh\left(a^2\right)\left[\sinh\left(|z|^2\right) + \left|\sinh\left(z^2\right)\right|\right] \tag{2.19}$$

for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

Now if in (2.19) we choose $z = ib$, with $b \in \mathbb{R}$, then we obtain the inequality

$$|\sin(iab)|^2 \le \sinh\left(a^2\right) \sinh\left(b^2\right) \tag{2.20}$$

for any $a, b \in \mathbb{R}$.

(2) The function

$$f(z) = \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad z \in \mathbb{C}$$

has the transform

$$f_A(z) = \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right) \quad \text{for } z \in \mathbb{C}.$$

Utilising the inequality (2.18) for f as above gives

$$|\cos(az)|^2 \le \frac{1}{2} \cosh\left(a^2\right)\left[\cosh\left(|z|^2\right) + \left|\cosh\left(z^2\right)\right|\right] \tag{2.21}$$

for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$. In particular, we have, with $z = ib$,

$$|\cos(iab)|^2 \le \cosh\left(a^2\right) \cosh\left(b^2\right) \tag{2.22}$$

for each $a, b \in \mathbb{R}$.

From a different perspective, we can state the following result which establishes a connection between two power series, one having positive coefficients.

Theorem 3. *Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be two power series with $b_n \in \mathbb{R}$ and $a_n > 0$ for $n \in \mathbb{N}$. If g is convergent on $D(0, R_1)$, f is convergent on $D(0, R_2)$ and the numerical series $\sum_{n=0}^{\infty} \frac{b_n^2}{a_n}$ is convergent, then we have the inequality:*

$$|g(z)|^2 \le \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{b_n^2}{a_n}\right)\left[f\left(|z|^2\right) + \left|f\left(z^2\right)\right|\right], \tag{2.23}$$

for any $z \in \mathbb{C}$ with $z \in D(0, R_1)$ and $a, |z|^2 \in D(0, R_2)$.

Proof. Utilising the de Bruijn weighted inequality (2.4) we can state that

$$\begin{aligned} |g(z)|^2 &= \left|\sum_{n=0}^{\infty} \frac{b_n}{a_n} a_n z^n\right|^2 \\ &\leq \frac{1}{2} \sum_{n=0}^{\infty} a_n \left(\frac{b_n}{a_n}\right)^2 \left[\sum_{n=0}^{\infty} a_n |z|^{2n} + \left|\sum_{n=0}^{\infty} a_n z^{2n}\right|\right] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{b_n^2}{a_n} \left[f\left(|z|^2\right) + \left|f\left(z^2\right)\right|\right] \end{aligned}$$

for $z \in \mathbb{C}$ with $z,\, z^2, |z|^2 \in D(0, R_1) \cap D(0, R_2)$. ∎

Remark 1. *The above inequality is useful in comparing different functions for which a bound for the numerical series* $\sum_{n=0}^{\infty} \frac{b_n^2}{a_n}$ *is known.*

Corollary 1. *Let* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be a power series with real coefficients and convergent on* $D(0, R)$. *If the numerical series* $\sum_{n=0}^{\infty} b_n^2$ *is convergent, then*

$$|g(z)|^2 \leq \frac{1}{2} \left(\sum_{n=0}^{\infty} b_n^2\right) \cdot \frac{\left|1 - z^2\right| + 1 - |z|^2}{|1 - z^2| \left(1 - |z|^2\right)} \tag{2.24}$$

for any $z \in \mathbb{C}$ *with* $z \in D(0, R)$ *and* $z^2, |z|^2 \in D(0, 1)$.

Corollary 2. *Let* $g(z)$ *be as in Corollary 1. If the numerical series* $\sum_{n=0}^{\infty} \left(n! b_n^2\right)$ *is convergent, then*

$$|g(z)|^2 \leq \frac{1}{2} \sum_{n=0}^{\infty} \left(n! b_n^2\right) \left[\exp |z|^2 + \left|\exp\left(z^2\right)\right|\right], \tag{2.25}$$

for any $z \in D(0, R)$.

Remark 2. *If we consider the series expansion*

$$\frac{1}{z} \cdot \ln \frac{1}{1 - z} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n, \qquad z \in D(0, 1) \setminus \{0\},$$

then, on utilising the inequality (2.24) for the choice $b_n := \frac{1}{n+1}$ *and taking into account that*

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$$

we can state the inequality

$$|\ln(1 - z)| \leq \frac{\pi^2}{12} |z| \frac{\left|1 - z^2\right| + 1 - |z|^2}{|1 - z^2| \left(1 - |z|^2\right)} \quad \textit{for any} \;\; z \in D(0, 1). \tag{2.26}$$

3. Some Inequalities for the Polylogarithm

Before we state our results for the polylogarithm that can be obtained on utilising the de Bruijn inequality, we recall some concepts that will be used in the sequel.

The *polylogarithm* $Li_n(z)$, also known as the *de Jonquières function* is the function defined by

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{3.1}$$

defined in the complex plane over the unit disk $D(0,1)$.

The special case $z = 1$ reduces to $Li_s(z) = \zeta(s)$, where ζ is the *Riemann zeta function.*

The polylogarithm of nonnegative integer order arises in the sums of the form

$$\sum_{k=1}^{\infty} k^n r^k = Li_{-n}(r) = \frac{1}{(1-r)^{n+1}} \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle k^{n-i}$$

where $\left\langle {n \atop i} \right\rangle$ is an *Eulerian number*, namely, we recall that

$$\left\langle {n \atop k} \right\rangle := \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{i} (k-j+1)^n .$$

Polylogarithms also arise in sums of generalised harmonic numbers $H_{n,r}$ as

$$\sum_{n=1}^{\infty} H_{n,r} z^n = \frac{Li_r(z)}{1-z} \quad \text{for} \quad z \in D(0,1),$$

where, we recall that

$$H_{n,r} := \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{and} \quad H_{n,1} := H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$Li_{-2}(z) = \frac{z(z+1)}{(1-z)^3}, \qquad Li_{-1}(z) = \frac{z}{(1-z)^2},$$
$$Li_0(z) = \frac{z}{1-z} \quad \text{and} \quad Li_1(z) = -\ln(1-z), \quad z \in D(0,1).$$

At argument $z = -1$, the general polylogarithms become $Li_n(-1) = -\eta(x)$, where $\eta(x)$ is the *Dirichlet eta function.*

It is obvious that Li_n being a power series with nonnegative coefficients all the results in the above section hold true. Therefore we have, for instance the inequality:

$$|Li_n(az)|^2 \le \frac{1}{2} Li_n(a^2) \left[Li_n\left(|z|^2\right) + \left|Li_n(z^2)\right| \right] \tag{3.2}$$

for any $a \in (-1,1)$ and $z \in D(0,1)$, where n is a negative or a positive integer.

In the following we present some results that connect difficult order polylogarithms:

Theorem 4. *Let $a \in (-1,1)$, $z \in D(0,1)$ and p, q, r integers such that the following series exist. Then*

$$|Li_{r+p+q}(az)|^2 \le \frac{1}{2} Li_{r+2p}(a^2) \left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}(z^2)\right| \right] \tag{3.3}$$

Proof. Utilising the de Bruijn's inequality with positive weights we have:

$$\begin{aligned}
\left|Li_{r+p+q}(az)\right|^2 &= \left|\sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{1}{k^p} a^k \cdot \frac{1}{k^q} z^k\right|^2 \\
&\le \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{a^{2k}}{k^{2p}}\right)\left[\sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{|z|^{2k}}{k^{2q}} + \left|\sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{z^{2k}}{k^{2q}}\right|\right] \\
&= \frac{1}{2} Li_{r+2p}\left(a^2\right)\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}\left(z^2\right)\right|\right]
\end{aligned}$$

and the inequality is proved. ∎

In particular, we can state the following inequality which incorporates the zeta function:

Corollary 3. *Let $z \in D(0,1)$ and p, q, r integers such that $r + 2p > 1$. Then*

$$\left|Li_{r+p+q}(z)\right|^2 \le \frac{1}{2}\zeta(r+2p)\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}\left(z^2\right)\right|\right] \tag{3.4}$$

The proof follows by Theorem 4 for $a = 1$.

Remark 3. *On utilising (3.4) and taking into account that some particular values of ζ are known, then we can state the following results:*

$$\left|Li_{q+1}(z)\right|^2 \le \frac{\pi^2}{12}\left[Li_{2q}\left(|z|^2\right) + \left|Li_{2q}\left(z^2\right)\right|\right] \tag{3.5}$$

and

$$\left|Li_{q+2}(z)\right|^2 \le \frac{\pi^4}{180}\left[Li_{2q}\left(|z|^2\right) + \left|Li_{2q}\left(z^2\right)\right|\right] \tag{3.6}$$

for any $z \in D(0,1)$ and q an integer.

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