SCHUR-CONVEXITY AND SCHUR-GEOMETRIC CONVEXITY FOR A CLASS OF THE MEANS

DA-MAO LI AND HUAN-NAN SHI

ABSTRACT. Two new means of two variables are defined by the hyperbolic and the arc-hyperbolic function composite function, and their Schur-convex and Schur-Geometric convex properties are discussed. As application, a new chain of inequalities is established.

1. Introduction

Question 11031 of the American Mathematical Monthly[1]: Define the monster mean $M(x,y)$ of two positive real number to be $\ln N(x,y)$, where $N(x,y)$ is the function

$$\frac{1+\ln\left(\sqrt{1+f(x,y)}+\sqrt{f(x,y)}\right)}{1-\ln\left(\sqrt{1+f(x,y)}+\sqrt{f(x,y)}\right)},$$

and

$$f(x,y)=\frac{\left(e^{2(e^x-1)\cdot(e^x+1)^{-1}}-1\right)\left(e^{2(e^y-1)\cdot(e^y+1)^{-1}}-1\right)}{4e^{\left((e^x-1)\cdot(e^x+1)^{-1}+(e^y-1)\cdot(e^y+1)^{-1}\right)}}.$$

Prove or disprove:

$$M(x,y)\le G(x,y)=\sqrt{xy}. \tag{1}$$

In [2, p.118-121], X.-M. Zhang proves that (1) holds. In [3], D.-M. Li and H.-N. Shi pointed out that $M(x,y)$ can be expressed in the composite form of the hyperbolic function and the arc-hyperbolic function as follows

$$M(x,y)=2\tanh^{-1}\sinh^{-1}\sqrt{\sinh(\tanh(x/2))\sinh(\tanh(y/2))}.$$

And by using geometric concavity of the hyperbolic function $\sinh(\tanh x)$ is given an anther proof of (1).

In this paper we shall define two new means that analogous to $M(x,y)$ as follows

$$D(x,y)=\sinh[(\sinh^{-1}x+\sinh^{-1}y)/2]$$

and

$$\bar{D}(x,y)=\sinh^{-1}[(\sinh x+\sinh y)/2]$$

2000 *Mathematics Subject Classification.* Primary 26D15, 26A51.

Key words and phrases. hyperbolic function, arc-hyperbolic function, mean of two variables, Schur-convexity, Schur- geometric convexity, inequality.

The second author was supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM200611417009).

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Our main aim is to discuss their Schur-convex and Schur-geometric convex properties and to establish a chain of inequalities for the mean values:

$$x \le M(x,y) \le M\left(\widetilde{u}(t),\widetilde{v}(t)\right) \le G(x,y) \le D\left(\widetilde{u}(t),\widetilde{v}(t)\right) \le D(x,y)$$
$$\le D\left(u(t),v(t)\right) \le A(x,y) \le \bar{D}(u(t),v(t)) \le \bar{D}(x,y) \le y.$$

where $0 < x < y, \frac{1}{2} \le t \le 1, u(t) = ty + (1-t)x, v(t) = tx + (1-t)y, \widetilde{u}(t) = b^t a^{1-t}, \widetilde{v}(t) = a^t b^{1-t}$. At the same time, a new proof of the problem no. 11031 is given.

2. Definitions and Lemmas

We denote the set of n-dimensional row vector on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n : x_i \ge 0, i = 1,\dots,n\},$$

$$\mathbb{R}^n_{++} = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n : x_i > 0, i = 1,\dots,n\}.$$

In particular, $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$ denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

We need the following definitions and lemmas.

Definition 1. [4, 5] Let $\boldsymbol{x} = (x_1,\dots,x_n)$ and $\boldsymbol{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k = 1,2,\dots,n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
(2) $\boldsymbol{x} \ge \boldsymbol{y}$ means $x_i \ge y_i$ for all $i = 1,2,\dots,n$. let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \ge \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \ge \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only $-\varphi$ is increasing.
(3) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only $-\varphi$ is Schur-convex function.

Definition 2. [2, 6] let $\Omega \subset \mathbb{R}^n_{++}$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1,\dots,\ln x_n) \prec (\ln y_1,\dots,\ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only $-\varphi$ is Schur-geometrically convex function.

Lemma 1 ([4, p. 7])**.** *A function $\varphi(\boldsymbol{x})$ is increasing if and only if $\nabla\varphi(\boldsymbol{x}) \ge 0$ for $\boldsymbol{x} \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, $\varphi: \Omega \to R$ is differentiable, and*

$$\nabla\varphi(\boldsymbol{x}) = \left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1},\dots,\frac{\partial\varphi(\boldsymbol{x})}{\partial x_n}\right) \in \mathbb{R}^n.$$

Lemma 2 ([4, p. 5])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior set. Ω^0 is the interior of Ω. $\varphi: \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0 (\le 0) \tag{2}$$

holds for any $\boldsymbol{x} = (x_1,x_2,\cdots,x_n) \in \Omega^0$.

Lemma 3 ([2, p. 108])**.** *Let $\Omega \subset \mathbb{R}_+^n$ is a symmetric and has a nonempty interior geometrically convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}_+$ is continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{3}$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is the Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 4. *Let $x \leq y, u(t) = ty + (1-t)x, v(t) = tx + (1-t)y$ and $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$, Then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (x, y). \tag{4}$$

Proof. it is easy to see that $u(t_1) \geq v(t_1), u(t_2) \geq v(t_2), y \geq u(t_1) \geq u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = x + y$. That is (4) is holds. □

From the Lemmas 4, we easily get

Lemma 5. *Let $0 < x \leq y, \widetilde{u}(t) = y^t x^{1-t}, \widetilde{v}(t) = x^t y^{1-t}$ and $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$. Then*

$$(\ln \widetilde{u}(t_2), \ln \widetilde{v}(t_2)) \prec (\ln \widetilde{u}(t_1), \ln \widetilde{v}(t_1)) \prec (\ln x, \ln y), \tag{5}$$

Lemma 6. [3] *Let $x \in \mathbb{R}$. Then*

$$\sinh(2 \tanh x) = \sum_{k=0}^{+\infty} \frac{(2 \tanh x)^{2k+1}}{(2k+1)!}. \tag{6}$$

Lemma 7. [3] *Let $x \in \mathbb{R}_+$. Then*

$$\tanh x \leq \sinh(\tanh x) \leq x \leq \sinh x \leq \sinh(\sinh x) \tag{7}$$

Lemma 8. *$g(x) = \left[\cosh^2 x \tanh(\tanh x)\right]^{-1}$ is a decreasing function on $\mathbb{R}_{++}$.*

Proof.

$$\begin{aligned} g'(x) &= -\frac{2 \sinh \cosh x \tanh(\tanh x) + \cosh^2 x [\cosh(\tanh x) \cosh x]^{-2}}{\cosh^4 x \tanh^2(\tanh x)} \\ &= -\frac{\sinh 2x \tanh(\tanh x) \cosh^2(\tanh x) + 1}{\cosh^4 x \tanh^2(\tanh x)} \leq 0, \end{aligned}$$

it follows that $g(x)$ is a decreasing function on $\mathbb{R}_{++}$. □

Lemma 9. *$h(x) = xg(x) = x\left[\cosh^2 x \tanh(\tanh x)\right]^{-1}$ is a decreasing function on $\mathbb{R}_{++}$.*

Proof.

$$h'(x) = \frac{q(x)}{\cosh^4 x \tanh^2(\tanh x)}$$

where

$$q(x) = \cosh^2 x \tanh(\tanh x) - x \sinh 2x \tanh(\tanh x) - x \cosh^2 x [\cosh(\tanh x) \cosh x]^{-2}.$$

We only need to prove $q(x) \leq 0$.

$$\begin{aligned}
q(x) &= (\cosh^2 x - x\sinh 2x)\tanh(\tanh x) - x[\cosh(\tanh x)]^{-2} \\
&= (\cosh^2 x - x\sinh 2x)\frac{\sinh(\tanh x)}{\cosh(\tanh x)} - x[\cosh(\tanh x)]^{-2} \\
&= (\cosh^2 x - x\sinh 2x)\frac{\sinh(\tanh x)\cosh(\tanh x)}{\cosh^2(\tanh x)} - x[\cosh(\tanh x)]^{-2} \\
&= (\cosh^2 x - x\sinh 2x)\left[\sinh(\tanh x)\cosh(\tanh x) - x\right][\cosh(\tanh x)]^{-2} \\
&= \frac{1}{2}\left[\sinh(2\tanh x)(\cosh^2 x - x\sinh 2x) - 2x\right][\cosh(\tanh x)]^{-2} \\
&= \frac{1}{2}p(x)[\cosh(\tanh x)]^{-2}
\end{aligned}$$

where

$$p(x) = \sinh(2\tanh x)\cosh^2 x(1 - 2x\tanh x) - 2x.$$

We only need to prove $p(x) \leq 0$.

$$\begin{aligned}
p(x) &\leq \sinh(2\tanh x)\cosh^2 x(1 - 2\tanh^2 x) - 2\tanh x(\because \tanh x < x) \\
&= \sinh(2\tanh x)(\cosh^2 x - 2\sinh^2 x) - 2\tanh x \\
&= \sinh(2\tanh x)\frac{\cosh^2 x - 2\sinh^2 x}{\cosh^2 x - \sinh^2 x} - 2\tanh x
\end{aligned}$$

(let $u = \tanh x$)

$$\begin{aligned}
&= \sinh(2u)\left(1 - \frac{u^2}{1-u^2}\right) - 2u \\
&= \sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!}\left(1 - u^2 - u^4 - \cdots - u^{2n} - \cdots\right) - 2u \\
&= \sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!}(1-u^2) - 2u - \sum_{k=2}^{+\infty}u^{2k}\sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} - \sum_{k=0}^{+\infty}\frac{2^{2k+1}u^{2k+3}}{(2k+1)!} - 2u - \sum_{k=2}^{+\infty}u^{2k}\sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=1}^{+\infty}\frac{2^{2k+1}u^{2k+1}}{(2k+1)!} - \sum_{k=0}^{+\infty}\frac{2^{2k+1}u^{2k+3}}{(2k+1)!} - \sum_{k=2}^{+\infty}u^{2k}\sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{+\infty}\frac{2^{2k+3}u^{2k+3}}{(2k+3)!} - \sum_{k=0}^{+\infty}\frac{2^{2k+1}u^{2k+3}}{(2k+1)!} - \sum_{k=2}^{+\infty}u^{2k}\sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{+\infty}\left[\frac{2^{2k+3} - 2^{2k+1}(2k+2)(2k+3)}{(2k+3)!}\right]u^{2k+3} - \sum_{k=2}^{+\infty}u^{2k}\sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} \\
&= -\sum_{k=0}^{+\infty}\left[\frac{2^{2k+1}(4k^2+10k+2)}{(2k+3)!}\right]u^{2k+3} - \sum_{k=2}^{+\infty}u^{2k}\sum_{k=0}^{+\infty}\frac{(2u)^{2k+1}}{(2k+1)!} \leq 0,
\end{aligned}$$

it follows that $h(x)$ is a decreasing function on $\mathbb{R}_{++}$. □

3. Main results and their proofs

In the following, we are in a position to state our main results and give proofs of them.

Theorem 1. *let $\Omega \subset \mathbb{R}^n_{++}$, $f : \Omega \to \mathbb{R}_+$ is increasing and differentiable. If f is a Schur-convex function, then f is also a Schur-geometrically convex function.*

Proof. Let $\boldsymbol{x} \in \Omega$. Since f is increasing, then $\nabla f(\boldsymbol{x}) = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \dots, \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right) \geq 0$. If f is a Schur-convex function, then $(x_1 - x_2) \left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) \geq 0$. And then

$$
\begin{aligned}
&(\ln x_1 - \ln x_2) \left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) \\
&= (\ln x_1 - \ln x_2) \left(x_1 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} + x_1 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_2} \right) \\
&= x_1 (\ln x_1 - \ln x_2) \left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) + \frac{\partial f}{\partial x_2} (x_1 - x_2) (\ln x_1 - \ln x_2) \geq 0.
\end{aligned}
$$

From the Lemmas 3, f is the Schur-geometrically convex function. □

Theorem 2. *let $\Omega \subset \mathbb{R}^n_{++}$, $f : \Omega \to \mathbb{R}_+$ is decreasing and differentiable. If f is a Schur-concave function, then f is also a Schur-geometrically concave function.*

The Theorem 2 can be proved in the same way as shown before.

Theorem 3. *$M(x, y)$ is a*

1. *increasing and Schur-concave function on $\mathbb{R}^2_+$;*
2. *Schur-geometrically concave function on $\mathbb{R}^2_+$.*

Proof. (1) By the computing, we get

$$
\frac{\partial M}{\partial x} = \frac{\cosh(\tanh(x/2)) \sinh(\tanh(y/2))}{F(x, y) \cosh^2(x/2)}, \quad \frac{\partial M}{\partial y} = \frac{\cosh(\tanh(y/2)) \sinh(\tanh(x/2))}{F(x, y) \cosh^2(y/2)}
$$

where

$$
\begin{aligned}
F(x, y) = 2 &\left[1 - \left(\sinh^{-1} \sqrt{\sinh(\tanh(x/2)) \sinh(\tanh(y/2))} \right)^2 \right] \\
&\cdot \sqrt{1 + \sinh(\tanh(x/2)) \sinh(\tanh(y/2))} \sqrt{\sinh(\tanh(x/2)) \sinh(\tanh(y/2))}
\end{aligned}
$$

From $0 < \tanh(t) < 1$, it follows that $0 < \sinh(\tanh t) < \sinh 1$. Since $\cosh(t) > 0$, $\sinh(t) > 0$ and $\tanh(t) > 0$ with $t > 0$, for $x > 0$ and $y > 0$, we have

$$
0 < \sinh(\tanh(x/2)) \sinh(\tanh(y/2)) < \sinh^2 1,
$$

and then

$$
1 - \left(\sinh^{-1} \sqrt{\sinh(\tanh(x/2)) \sinh(\tanh(y/2))} \right)^2 > 0.
$$

So that $F(x,y)>0$. It implies that $\frac{\partial M}{\partial x}\geq 0$, $\frac{\partial M}{\partial y}\geq 0$. From Lemma 1, $M(x,y)$ is a increasing function on $\mathbb{R}_+^2$.

$$\begin{aligned}\triangle :&= (x-y)\left(\frac{\partial M}{\partial x}-\frac{\partial M}{\partial y}\right)\\ &=\frac{x-y}{F(x,y)}\left[\frac{\cosh(\tanh(x/2))\sinh(\tanh(y/2))}{\cosh^2(x/2)}-\frac{\cosh(\tanh(y/2))\sinh(\tanh(x/2))}{\cosh^2(y/2)}\right]\\ &=\frac{(x-y)\sinh(\tanh(x/2))\sinh(\tanh(y/2))}{F(x,y)}\\ &\cdot\left[\frac{\cosh(\tanh(x/2))}{\cosh^2(x/2)\sinh(\tanh(x/2))}-\frac{\cosh(\tanh(y/2))}{\cosh^2(y/2)\sinh(\tanh(y/2))}\right]\\ &=\frac{\sinh(\tanh(x/2))\sinh(\tanh(y/2))}{F(x,y)}(x-y)\left[g(x)-g(y)\right].\end{aligned}$$

From Lemma 8, $g(x)=\left[\cosh^2 x\tanh(\tanh x)\right]^{-1}$ is a decreasing function on $\mathbb{R}_{++}$. So that $(x-y)\left[g(x)-g(y)\right]\leq 0$, and then $\triangle\leq 0$. From Lemma 2, $M(x,y)$ is a Schur-concave function on $\mathbb{R}_+^2$.

(2)

$$\begin{aligned}\Lambda :&= (\ln x-\ln y)\left(x\frac{\partial M}{\partial x}-y\frac{\partial M}{\partial y}\right)\\ &=\frac{(\ln x-\ln y)}{F(x,y)}\left[\frac{x\cosh(\tanh(x/2))\sinh(\tanh(y/2))}{\cosh^2(x/2)}-\frac{y\cosh(\tanh(y/2))\sinh(\tanh(x/2))}{\cosh^2(y/2)}\right]\\ &=\frac{(\ln x-\ln y)\sinh(\tanh(x/2))\sinh(\tanh(y/2))}{F(x,y)}\\ &\cdot\left[\frac{x\cosh(\tanh(x/2))}{\cosh^2(x/2)\sinh(\tanh(x/2))}-\frac{y\cosh(\tanh(y/2))}{\cosh^2(y/2)\sinh(\tanh(y/2))}\right]\\ &=\frac{\sinh(\tanh(x/2))\sinh(\tanh(y/2))}{F(x,y)}(\ln x-\ln y)\left[xg(x)-yg(y)\right].\end{aligned}$$

From Lemma 9, $h(x)=xg(x)=x\left[\cosh^2 x\tanh(\tanh x)\right]^{-1}$ is decreasing on $\mathbb{R}_{++}$ and $\ln x$ is increasing on $\mathbb{R}_{++}$. So that $(\ln x-\ln y)\left[xg(x)-yg(y)\right]\leq 0$, and then $\Lambda\leq 0$. From Lemma 3, $M(x,y)$ is a Schur-geometrically concave function on $\mathbb{R}_+^2$. □

Theorem 4. $D(x,y)=\sinh[(\sinh^{-1}x+\sinh^{-1}y)/2]$ *is a*

1. *increasing and Schur-concave function on* $\mathbb{R}_+^2$*;*
2. *Schur-geometrically convex function on* $\mathbb{R}_+^2$*.*

Proof. (1) From $\left(\sinh^{-1}t\right)'=\left(1+t^2\right)^{-1/2}$ and $(\sinh t)'=\cosh t$, it is deduced that

$$\frac{\partial D}{\partial x}=\frac{\cosh\left[\left(\sinh^{-1}x+(\sinh^{-1}y\right)/2\right]}{2\sqrt{1+x^2}}\geq 0,\ \frac{\partial D}{\partial x}=\frac{\cosh\left[\left(\sinh^{-1}x+(\sinh^{-1}y\right)/2\right]}{2\sqrt{1+y^2}}\geq 0.$$

From Lemma 1, $D(x,y)$ is a increasing function on $\mathbb{R}_+^2$.

$$\begin{aligned}
\triangle :&= (x-y)\left(\frac{\partial D}{\partial x}-\frac{\partial D}{\partial y}\right)\\
&= (x-y)\left\{\frac{\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]}{2\sqrt{1+x^2}}-\frac{\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]}{2\sqrt{1+y^2}}\right\}\\
&= (x-y)\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]\left(\frac{1}{2\sqrt{1+x^2}}-\frac{1}{2\sqrt{1+y^2}}\right)\\
&= (x-y)\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]\cdot\frac{\sqrt{1+y^2}-\sqrt{1+x^2}}{2\sqrt{1+x^2}\sqrt{1+y^2}}\\
&= (x-y)\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]\cdot\frac{y^2-x^2}{2\sqrt{1+x^2}\sqrt{1+y^2}\left(\sqrt{1+x^2}+\sqrt{1+y^2}\right)}\\
&= -\frac{(x-y)^2(x+y)\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]}{2\sqrt{1+x^2}\sqrt{1+y^2}\left(\sqrt{1+x^2}+\sqrt{1+y^2}\right)}\leq 0.
\end{aligned}$$

From Lemma 2, $D(x,y)$ is a Schur-concave function on $\mathbb{R}_+^2$.

(2)

$$\begin{aligned}
\Lambda :&= (\ln x-\ln y)\left(x\frac{\partial D}{\partial x}-y\frac{\partial D}{\partial y}\right)\\
&= (\ln x-\ln y)\left\{\frac{x\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]}{2\sqrt{1+x^2}}-\frac{y\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]}{2\sqrt{1+y^2}}\right\}\\
&= \frac{1}{2}(\ln x-\ln y)\cosh\left[\left(\sinh^{-1}x+\sinh^{-1}y\right)/2\right]\left(\frac{x}{\sqrt{1+x^2}}-\frac{y}{\sqrt{1+y^2}}\right)
\end{aligned}$$

From $\left(\frac{t}{\sqrt{1+t^2}}\right)^{'}=\frac{1}{(1+t^2)\sqrt{1+t^2}}\geq 0$, it is follows that $\frac{t}{\sqrt{1+t^2}}$ is a decreasing function on $\mathbb{R}_+$ and $\ln x$ is also increasing on $\mathbb{R}_{++}$. So that $(\ln x-\ln y)\left(\frac{x}{\sqrt{1+x^2}}-\frac{y}{\sqrt{1+y^2}}\right)\geq 0$, and then $\Lambda\geq 0$. From Lemma 3, $D(x,y)$ is a Schur-geometrically convex function on R_+^2. □

Theorem 5. $\bar{D}(x,y)=\sinh^{-1}[(\sinh x+\sinh y)/2]$ *is a*

(1) *increasing and Schur-convex function on* $\mathbb{R}_+^2$*;*
(2) *Schur-geometrically convex function on* $\mathbb{R}_+^2$*.*

Proof. (1) From $\left(\sinh^{-1}t\right)^{'}=\left(1+t^2\right)^{-1/2}$ and $\left(\sinh t\right)^{'}=\cosh t$, it is deduced that

$$\frac{\partial\bar{D}}{\partial x}=\frac{\cosh x}{2\sqrt{1+[(\sinh x+\sinh y)/2]^2}}\geq 0,\ \frac{\partial\bar{D}}{\partial x}=\frac{\cosh y}{2\sqrt{1+[(\sinh x+\sinh y)/2]^2}}\geq 0.$$

From Lemma 1, $\bar{D}(x,y)$ is a increasing function on $\mathbb{R}_+^2$.

$$\triangle := (x-y)\left(\frac{\partial\bar{D}}{\partial x}-\frac{\partial\bar{D}}{\partial y}\right)=\frac{(x-y)(\cosh x-\cosh y)}{2\sqrt{1+[(\sinh x+\sinh y)/2]^2}}$$

Since $\cosh x$ is a increasing function on R_+^2, $(x-y)(\cosh x - \cosh y) \geq 0$. And then $\triangle \geq 0$. From Lemma 2, $\bar{D}(x,y)$ is a Schur-convex function on $\mathbb{R}_+^2$.

(2) From (1) of Theorem 1, we know that $\bar{D}(x,y)$ is a Schur-geometrically convex function on $\mathbb{R}_+^2$. □

4. Applications

Theorem 6. *Let* $0 < x < y, u(t) = ty + (1-t)x, v(t) = tx + (1-t)y, \widetilde{u}(t) = b^t a^{1-t}, \widetilde{v}(t) = a^t b^{1-t}$ *and* $\frac{1}{2} \leq t \leq 1$. *Then*

$$x \leq M(x,y) \leq M\left(\widetilde{u}(t), \widetilde{v}(t)\right) \leq G(x,y) \leq D\left(\widetilde{u}(t), \widetilde{v}(t)\right) \leq D(x,y)$$
$$\leq D\left(u(t), v(t)\right) \leq A(x,y) \leq \bar{D}(u(t), v(t)) \leq \bar{D}(x,y) \leq y. \quad (8)$$

Proof. Since $\sinh(\tanh x) > 0$ with $x > 0$, $M(x,x) = x$. From $x < y$, it is follows that $(x,x) \leq (x,y)$. And then From (1) of Theorem 3, we get

$$x = M(x,x) \leq M(x,y).$$

From Lemma 5, we know that

$$(\ln\sqrt{xy}, \ln\sqrt{xy}) \prec (\ln\widetilde{u}(t), \ln\widetilde{v}(t)) \prec (\ln x, \ln y),$$

hence from (2) of Theorem 3 , we get

$$M(x,y) \leq M\left(\widetilde{u}(t), \widetilde{v}(t)\right) \leq M\left(\sqrt{xy}, \sqrt{xy}\right) = \sqrt{xy} = G(x,y),$$

and from (2) of Theorem 4, we get

$$G(x,y) = \sqrt{xy} = D\left(\sqrt{xy}, \sqrt{xy}\right) \leq D\left(\widetilde{u}(t), \widetilde{v}(t)\right) \leq D(x,y).$$

From Lemma 4, we know that

$$((x+y)/2, (x+y)/2) \prec (u(t), v(t)) \prec (x,y)\,,$$

hence from (1) of theorem 4, we have

$$D(x,y) \leq D(u(t), v(t)) \leq D\left((x+y)/2, (x+y)/2\right) = (x+y)/2 = A(x,y),$$

and from (1) of theorem 5, we get

$$A(x,y) = (x+y)/2 = \bar{D}((x+y)/2, (x+y)/2) \leq \bar{D}(u(t), v(t)) \leq \bar{D}(x,y).$$

Finally, since $(x,y) \leq (y,y)$, from (1) of theorem 5, we get

$$\bar{D}(x,y) \leq \bar{D}(y,y) = y.$$

The proof of Theorem 6 is be completed. □

Acknowledgements. The authors are indebted to professors Feng Qi for their many helpful and valuable comments and suggestions.

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