OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR THE STIELTJES INTEGRAL WITH LIPSCHITZIAN INTEGRANDS OR INTEGRATORS

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Abstract. Some Ostrowski and trapezoid type inequalities for the Stieltjes integral in the case of Lischitzian integrators for both Hölder continuous and monotoonic integrals are obtained. The dual case is also analysed. Applications for the midpoint rule are pointed out as well.

1. Introduction

The problem of approximating the Stieltjes integral $\int_a^b f(t)\,du(t)$ by the quantity $f(x)[u(b)-u(a)]$, which is a natural generalisation of the Ostrowski problem analysed in 1937 (see [1]), was apparently first considered in the literature by S.S. Dragomir in 2000 (see [2]) where he obtained the following result:

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{1.1}$$

$$\leq H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right]$$

$$\leq H\times\begin{cases}[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]; \\ [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f).\end{cases}$$

for each $x\in[a,b]$, provided f is of bounded variation on $[a,b]$ while $u:[a,b]\to\mathbb{R}$ is $r-H-$Hölder continuous, i.e., we recall that:

$$|u(x)-u(y)|\leq H|x-y|^r \quad \text{for each } x,y\in[a,b]. \tag{1.2}$$

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The dual case, i.e., when the *integrand* f is $q-K-$Hölder continuous and the integrator u is of bounded variation can be stated as [3]

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u) \tag{1.3}$$

for each $x\in[a,b]$.

The above inequalities provide, as important consequences, the following mid-point inequalities:

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \leq \begin{cases} \frac{(b-a)^r H}{2^r}\bigvee_a^b(f) \\ \\ \frac{(b-a)^q K}{2^q}\bigvee_a^b(u) \end{cases} \tag{1.4}$$

which can be numerically implemented and provide a quadrature rule for approximating the Stieltjes integral $\int_a^b f(t)\,du(t)$.

From a different view point, the authors of [4] considered the problem of approximating the Stieltjes integral $\int_a^b f(t)\,d(t)$ by the generalised trapezoid rule $u(b)[f(b)-f(x)]+u(a)[f(x)-f(a)]$ obtaining the result:

$$\left|\int_a^b f(t)\,du(t)-\{u(b)[f(b)-f(x)]+u(a)[f(x)-f(a)]\}\right| \leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \qquad x\in[a,b], \tag{1.5}$$

provided f is of bounded variation while u is of the $r-H-$Hölder type.

The dual case of (1.5), i.e., when f is of the $q-K-$Hölder type and u is of bounded variation was considered in [5] in which the authors obtained the inequality:

$$\begin{aligned} &\left|\int_a^b f(t)\,du(t)-\{u(b)[f(b)-f(x)]+u(a)[f(x)-f(a)]\}\right| \\ &\leq K\left[(x-a)^q\bigvee_a^x(u)+(b-x)^q\bigvee_x^b(u)\right] \\ &\leq \begin{cases} K\left[(x-a)^q+(b-x)^q\right]\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]; \\ \\ K\left[(x-a)^{q\alpha}+(b-x)^{q\alpha}\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{b}} \\ \qquad\qquad \text{if } \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ K\left[\frac{1}{2}|x-a|+\left|x-\frac{a+b}{2}\right|\right]^q\bigvee_a^b(u) \end{cases} \end{aligned} \tag{1.6}$$

for each $x\in[a,b]$.

The aim of the present paper is to establish new inequalities of the Ostrowski type, and, equivalently (see Theorem 1) for the generalised trapezoid rule, in the case of one Lipschitzian and the other a Hölder continuous function. The case where a function is monotonic nondecreasing is also investigated. The particular instance of the midpoint inequality is also analysed. Connections with earlier results for the Riemann integral are also pointed out.

2. The Case of Hölder Continuous and Lipschitzian Functions

The following result may be stated.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a $r - H-$Hölder continuous function on $[a, b]$, i.e.,*

$$|f(x) - f(y)| \leq H |x - y|^r \qquad \text{for any } x, y \in [a, b], \tag{2.1}$$

where $r \in (0, 1]$ and $H > 0$ are given, and $u : [a, b] \to \mathbb{R}$ is an $L-$Lipschitzian function on $[a, b]$, i.e.,

$$|u(x) - u(y)| \leq L |x - y|^r \qquad \text{for any } x, y \in [a, b], \tag{2.2}$$

then for any $x \in [a, b]$,

$$\left| [u(b) - u(a)] f(x) - \int_a^b f(t) \, du(t) \right| \leq \frac{LH}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right], \tag{2.3}$$

or, equivalently,

$$\left| \int_a^b u(t) \, df(t) - \{ u(b) [f(b) - f(x)] + u(a) [f(x) - f(a)] \} \right| \leq \frac{LH}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right]. \tag{2.4}$$

Proof. Note that if $p : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and $v : [a, b] \to \mathbb{R}$ is $L-$Lipschitzian, then the Stieltjes integral $\int_a^b p(t) \, dv(t)$ exists and

$$\left| \int_a^b p(t) \, dv(t) \right| \leq L \int_a^b |p(t)| \, dt. \tag{2.5}$$

Utilising this property,

$$\begin{aligned} \left| [u(b) - u(a)] f(x) - \int_a^b f(t) \, du(t) \right| &= \left| \int_a^b [f(x) - f(t)] \, du(t) \right| \\ &\leq L \int_a^b |f(x) - f(t)| \, dt \\ &\leq LH \int_a^b |x - t|^r \, dt \\ &= \frac{LH}{r+1} \left[(x-a)^{r+1} + (b-x)^{r+1} \right], \end{aligned}$$

and the inequality (2.3) is proved.

Since, by the integration by parts formula for Stieltjes integrals we have,

$$\begin{aligned} &[u(b) - u(a)] f(x) - \int_a^b f(t) \, du(t) \\ &\qquad = \int_a^b u(t) \, df(t) - u(b) [f(b) - f(x)] - u(a) [f(x) - f(a)], \end{aligned}$$

then hence (2.4) is a direct consequence of (2.3). ∎

Remark 1. *If f is assumed to be $K-$Lipschitzian, then from (2.3) and (2.4) we get the equivalent inequalities:*

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \le HL\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 \tag{2.6}$$

and

$$\left|\int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right| \le HL\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2, \tag{2.7}$$

for each $x\in[a,b]$.

The midpoint inequality is useful for numerical implementation and is incorporated in the following corollary.

Corollary 1. *With the assumptions of Theorem 1,*

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \le \frac{1}{2^r(r+1)}LH(b-a)^{r+1}, \tag{2.8}$$

and

$$\left|\int_a^b u(t)\,df(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right| \le \frac{1}{2^r(r+1)}LH(b-a)^{r+1} \tag{2.9}$$

respectively.

Remark 2. *If $u(t)=t$ in the above, then the results for the Riemann integral obtained in* [6] *are recaptured.*

Remark 3. *In terms of probability density functions, if $w:[a,b]\to[0,\infty)$ is such that $w\in L_\infty[a,b]$, i.e., $\|w\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|w(t)|<\infty$, and $\int_a^b w(s)\,ds=1$, then the function $u(t)=\int_a^t w(s)\,ds$ is $L-$Lipschitzian with the constant $L=\|w\|_{[a,b],\infty}$ and the inequalities (2.3) and (2.4) can be written as:*

$$\left|f(x)-\int_a^b w(t)f(t)\,dt\right| \le \frac{H\|w\|_{[a,b],\infty}}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right] \tag{2.10}$$

and

$$\left|\int_a^b\left(\int_a^t w(s)\,ds\right)df(t)-f(b)-f(x)\right| \le \frac{H\|w\|_{[a,b],\infty}}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right] \tag{2.11}$$

for any $x\in[a,b]$.

The dual case, i.e., when f is Lipschitzian and u is Hölder continuous admits some slight variations as follows.

Theorem 2. *Let $x \in [a,b]$ and assume that f is L_1-Lipschitzian on the interval $[a,x]$ and L_2-Lipschitzian on the interval $[a,b]$ $(L_1, L_2 > 0)$ while the function $u : [a,b] \to \mathbb{R}$ satisfies some local Hölder conditions (properties), namely,*

$$|u(t) - u(a)| \leq H_1 |t-a|^{\alpha_1} \qquad \text{for any } \ t \in [a,x] \tag{2.12}$$

and

$$|u(b) - u(t)| \leq H_2 |t-b|^{\alpha_2} \qquad \text{for any } \ t \in [x,b] \tag{2.13}$$

where $H_1, H_2 > 0$, $\alpha_1, \alpha_2 \in (-1, \infty)$ (notice the difference for α_1, α_2), then,

$$\begin{aligned} &\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \\ &\qquad \leq \frac{L_1 H_1 (x-a)^{\alpha_1+1}}{\alpha_1+1} + \frac{L_2 H_2 (b-x)^{\alpha_2+1}}{\alpha_2+1} \end{aligned} \tag{2.14}$$

or, equivalently,

$$\begin{aligned} &\left| \int_a^b u(t)\, df(t) - u(b)[f(b) - f(x)] - u(a)[f(x) - f(a)] \right| \\ &\qquad \leq \frac{L_1 H_1 (x-a)^{\alpha_1+1}}{\alpha_1+1} + \frac{L_2 H_2 (b-x)^{\alpha_2+1}}{\alpha_2+1}. \end{aligned} \tag{2.15}$$

Proof. We use the following generalisation of the *Montgomery identity* for the Stieltjes integral established by S.S. Dragomir in [2]:

$$\begin{aligned} &[u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \\ &\qquad = \int_a^x [u(t) - u(a)]\, df(t) + \int_x^b [u(t) - u(b)]\, df(t) \end{aligned} \tag{2.16}$$

for any $x \in [a,b]$.

Taking the modulus we have

$$\begin{aligned} &\left| [u(b) - u(a)] f(x) - \int_a^b f(t)\, du(t) \right| \\ &\leq \left| \int_a^x [u(t) - u(a)]\, df(t) \right| + \left| \int_x^b [u(t) - u(b)]\, df(t) \right| \\ &\leq L_1 \int_a^x |u(t) - u(a)|\, dt + L_2 \int_x^b |u(t) - u(b)|\, dt \\ &\leq H_1 L_1 \int_a^x (t-a)^{\alpha_1}\, dt + H_2 L_2 \int_x^b (b-x)^{\alpha_2}\, dt \\ &= \frac{H_1 L_1 (x-a)^{\alpha_1+1}}{\alpha_1+1} + \frac{H_2 L_2 (b-x)^{\alpha_2+1}}{\alpha_2+1}, \end{aligned}$$

and the inequality (2.14) is obtained. ∎

Remark 4. *It is obvious that, if we assume that f is $K-$Lipschitzian on the whole interval $[a,b]$ while u is of the $q-$Hölder type with $q \in (0,1]$, then from Theorem 2 we can obtain the following inequality which is the dual of (2.3):*

$$\left| [u(b)-u(a)] f(x) - \int_a^b f(t)\,du(t) \right| \le \frac{KH}{q+1}\left[(x-a)^{q+1} + (b-x)^{q+1} \right] \tag{2.17}$$

for any $x \in [a,b]$.

Remark 5. *From the tools utilised in the proofs of Theorem 1 and 2, one can easily realise that if in the first result it is natural to assume the global property of $r-H-$Hölder continuity for the integrand and $L-$Lipschitzian property for the integrator, then in the second theorem the local properties around the end-points a and b qualify as natural as well. Moreover, we observe that in (2.4) the order of approximation is* $\min(\alpha_1,\alpha_2)+1$ *which can be higher than the order of approximation in (2.3) which is $r+1$ (maximum 2 for $r=1$). However, this can be improved if some local conditions around $x\in[a,b]$ are assumed.*

If u is T_1-Lipschitzian on $[a,x]$ and T_2-Lipschitzian on $[x,b]$ and the function f satisfies around x the following conditions

$$|f(t)-f(x)| \le V_1 |t-x|^{\beta_1}, \qquad t\in[a,x],$$

and

$$|f(t)-f(x)| \le V_2 |t-x|^{\beta_2}, \qquad t\in[x,b],$$

where $V_1,V_2>0$, $\beta_1,\beta_2\in(-1,\infty)$ are given, then, following the proof of Theorem 1, we have,

$$\begin{aligned}
&\left| [u(b)-u(a)] f(x) - \int_a^b f(t)\,du(t) \right| \\
&= \left| \int_a^x (f(x)-f(t))\,du(t) + \int_x^b (f(x)-f(t))\,du(t) \right| \\
&\le \left| \int_a^x (f(x)-f(t))\,du(t) \right| + \left| \int_x^b (f(x)-f(t))\,du(t) \right| \\
&\le T_1 \int_a^x |f(x)-f(t)|\,dt + T_2 \int_x^b |f(x)-f(t)|\,dt \\
&\le \frac{T_1 V_1 (x-a)^{\beta_1+1}}{\beta_1+1} + \frac{T_2 V_2 (b-x)^{\beta_2+1}}{\beta_2+1}
\end{aligned}$$

giving a similar result to the one in Theorem 2.

3. The Case of Monotonic and Lipschitzian Functions

The case where the integrator in monotonic nondecreasing is incorporated in the following result:

Theorem 3. *Let $x\in[a,b]$ and assume that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,x]$ and $[x,b]$ (it may not be monotonic nondecreasing on the whole* *of $[a,b]$). If u is L_1-Lipschitzian on $[a,x]$ and L_2-Lipschitzian on $[x,b]$, then,*

$$
\begin{aligned}
&\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \\
&\leq L_2\int_x^b f(t)\,dt - L_1\int_a^x f(t)\,dt - [L_2(b-x)-L_1(x-a)]f(x) \\
&\leq L_2(b-x)[f(b)-f(x)] + L_1(x-a)[f(x)-f(a)] \\
&\leq \max\{L_1,L_2\}((b-x)[f(b)-f(x)]+(x-a)[f(x)-f(a)]) \\
&\leq \max\{L_1,L_2\}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)]; \\ \left[\frac{1}{2}[f(b)-f(a)]+\frac{1}{2}\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right](b-a)\end{cases}
\end{aligned}
\tag{3.1}
$$

and a similar inequality holds for the generalised trapezoid rule.

Proof. As in the proof of Theorem 1 above, we have,

$$
\begin{aligned}
&\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \\
&\leq L_1\int_a^x |f(x)-f(t)|\,dt + L_2\int_x^b |f(x)-f(t)|\,dt \\
&= L_1(x-a)f(x) - L_1\int_a^x f(t)\,dt + L_2\int_x^b f(t)\,dt - L_2(b-x)f(x) \\
&= L_2\int_x^b f(t)\,dt - L_1\int_a^x f(t)\,dt - [L_2(b-x)-L_1(x-a)]f(x)
\end{aligned}
$$

proving the first inequality in (3.1).

Now, on utilising the monotonicity property of f on both intervals, we have:

$$
\int_x^b f(t)\,dt \leq (b-x)f(b) \quad \text{and} \quad \int_a^x f(t)\,dt \geq (x-a)f(a)
$$

which implies that,

$$
\begin{aligned}
&L_2\int_x^b f(t)\,dt - L_1\int_a^x f(t)\,dt - [L_2(b-x)-L_1(x-a)]f(x) \\
&\leq L_2(b-x)f(b) - L_1(x-a)f(a) - [L_2(b-x)-L_1(x-a)]f(x) \\
&= L_2(b-x)[f(b)-f(x)] + L_1(x-a)[f(x)-f(a)],
\end{aligned}
$$

i.e., the second inequality in (3.1).

The last part is obvious by the property of the max function and we omit the details. ∎

Corollary 2. *If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ and u is L_1-Lipschitzian on the first interval and L_2-Lipschitzian on the second,* *then*

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{3.2}$$
$$\le L_2\int_{\frac{a+b}{2}}^b f(t)\,dt-L_1\int_a^{\frac{a+b}{2}} f(t)\,dt-\frac{b-a}{2}(L_2-L_1)f\left(\frac{a+b}{2}\right)$$
$$\le \frac{b-a}{2}\left[L_2[f(b)-f(x)]+L_1[f(x)-f(a)]\right]$$
$$\le \frac{b-a}{2}\max\{L_1,L_2\}[f(b)-f(a)].$$

Remark 6. *The case $u(t)=t$ (therefore $L_1=L_2=1$) retrieves the results obtained earlier for the Riemann integral in* [7].

The dual case is incorporated in the following result:

Theorem 4. *Let $x\in[a,b]$ and assume that u is monotonic nondecreasing on both $[a,x]$ and $[x,b]$, then,*

$$\left|[u(b)-u(x)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{3.3}$$
$$\le L_2(b-x)u(b)+L_1(x-a)u(a)+L_1\int_a^x u(t)\,dt-L_2\int_x^b u(t)\,dt$$
$$\le L_1(x-a)(u(x)-u(a))+L_2(b-x)(u(b)-u(x))$$
$$\le \max\{L_1,L_2\}\left[(x-a)(u(x)-u(a))+(b-x)(u(b)-u(x))\right]$$
$$\le \max\{L_1,L_2\}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][u(b)-u(a)];\\[2mm] \left[\frac{1}{2}[u(b)-u(a)]+\frac{1}{2}\left|u(x)-\frac{u(a)+u(b)}{2}\right|\right](b-a),\end{cases}$$

and a similar inequality holds for the generalised trapezoid rule.

Proof. As in the proof of Theorem 2 above, we have,

$$\left|[u(b)-u(x)]f(x)-\int_a^b f(t)\,du(t)\right|$$
$$\le L_1\int_a^x |u(t)-u(a)|\,dt+L_2\int_x^b|u(t)-u(b)|\,dt$$
$$=L_1\int_a^x u(t)\,dt-L_1(x-a)u(a)+L_2(b-x)u(b)-L_2\int_x^b u(t)\,dt$$

and the first inequality in (3.3) is proved.

By the monotonicity of u in both intervals $[a,x]$ and $[x,b]$ we have,

$$\int_a^x u(t)\,dt\le(x-a)u(x)\quad\text{and}\quad\int_x^b u(t)\,dt\ge(b-x)u(x)$$

which gives

$$\begin{aligned}
&L_1 \int_a^x u(t)\,dt - L_1(x-a)\,u(a) + L_2(b-x)\,u(b) - L_2 \int_x^b u(t)\,dt \\
&\leq L_1(x-a)\,u(x) - L_1(x-a)\,u(a) + L_2(b-x)\,u(b) - L_2(b-x)\,u(x) \\
&= L_1(x-a)\,[u(x)-u(a)] + L_2(b-x)\,[u(b)-u(x)]
\end{aligned}$$

and the second part of (3.3) also holds.

The last part is obvious and the details are omitted. ∎

Corollary 3. *If u is monotonic on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ while f is L_1-Lipschitzian on the first interval and L_2-Lipschitzian on the second, then,*

$$\begin{aligned}
&\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t)\right| \\
&\leq \frac{b-a}{2}\,[L_2 u(b) - L_1 u(a)] + L_1 \int_a^{\frac{a+b}{2}} u(t)\,dt - L_2 \int_{\frac{a+b}{2}}^b u(t)\,dt \\
&\leq \frac{b-a}{2}\left\{L_1\left[u\left(\frac{a+b}{2}\right) - u(a)\right] + L_2\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right\} \\
&\leq \frac{b-a}{2}\max\{L_1, L_2\}\,[u(b)-u(a)]\,.
\end{aligned} \tag{3.4}$$

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