SCHUR-CONVEXITY OF A MEAN OF CONVEX FUNCTION

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ABSTRACT. The Schur-convexity on the upper and the lower limit of the integral for a mean of the convex function is researched. As applications, a generalized logarithmic mean with a parameter is obtained and a relevant double inequality that is a extension of the known inequality is established.

1. INTRODUCTION

Let f be a convex function defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$ of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known as the Hadamard's inequality for convex function.

In [1], S. S. Dragomir established the following two theorems which are refinements of the first inequality of (1).

Theorem A ([1]). *If $f : [a, b] \to \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\mathrm{d}x$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \tag{2}$$

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Theorem B ([1]). *If $f : [a, b] \to \mathbb{R}$ is a convex function, and F is defined on $[0, 1]$ by*

$$F(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(tx + (1-t)y\right) \mathrm{d}x \, \mathrm{d}y$$

then

(i) F is convex on $[0, 1]$, symmetric about $\frac{1}{2}$, (i.e.$F(t) = F(1-t)$ for all $t \in [0, 1]$), F is increasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, and for all $t \in [0, 1]$, we have

$$F(t) \leq F(1) = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \tag{3}$$

and

$$F(t) \geq F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) \mathrm{d}x \, \mathrm{d}y \geq f\left(\frac{a+b}{2}\right) \tag{4}$$

(ii) for all $t \in [0, 1]$, we have

$$F(t) \geq max\{H(t), H(1-t)\}. \tag{5}$$

where $H(t)$ is defined in Theorem A.

In [2], S. S. Dragomir established the following theorem which is a extension of the relevant conclusion in [3].

Theorem C ([2]). *If $f : [a, b] \to \mathbb{R}$ is a convex function, and G is defined on $[0, 1]$ by*

$$G(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(ta + (1-t)x\right) + f\left(tb + (1-t)x\right)\right] \mathrm{d}x$$

then G is convex on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$\frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x = G(0) \leq G(t) \leq G(1) = \frac{f(a) + f(b)}{2}. \tag{6}$$

Remark 1. If f is concave, then (6) is reversed. (notice that $-f$ is convex)

In [4], N. Elezovic and J. Pecaric researched the Schur-convexity on the upper and the lower limit of the integral for the mean of the convex function and established the following important result by using the Hadamard's inequality.

Theorem D ([4]). *Let I be an interval with nonempty interior on $\mathbb{R}$ and f be a continuous function on I. Then*

$$\Phi(a, b) = \begin{cases} \frac{1}{b-a} \int_a^b f(t) dt, \ a, b \in I, \ a \neq b \\ f(a), \ a = b \end{cases}$$

is $Schur - convex(Schur - concave)$ *on* I^2 *if and if* f *is* $convex(concave) on I$.

In this paper, the first aim is to prove once again the inequality in (2) by Theorem D. The second aim is to establish the following results which are similar to Theorem D. As applications, a generalized logarithmic mean with a parameter is obtained and a relevant double inequality which is a extension of the known inequality is established.

Theorem 1. *Let* I *be an interval with nonempty interior on* $\mathbb{R}$ *and define a function of two variables as follows*

$$P(a,b) = \begin{cases} G(t), \ a, b \in I, \ a \neq b \\ f(a), \ a = b \end{cases}$$

(i) for $\frac{1}{2} \leq t \leq 1$*, if* f *is convex on* I*, then* $P(a,b)$ *is Schur-convex on* I^2*.*

(ii) for $0 \leq t \leq \frac{1}{2}$*, if* f *is concave on* I*, then* $P(a,b)$ *is Schur-concave on* I^2*.*

Theorem 2. *Let* I *be an interval with nonempty interior on* $\mathbb{R}$ *and* f *be a continuous function on* I*. For any* $t \in [0,1]$*, we define a function of two variables as follows*

$$Q(a,b) = \begin{cases} F(t), \ a, b \in I, \ a \neq b \\ f(a), \ a = b \end{cases}$$

if f *is convex (concave) on* I*, then* $P(a,b)$ *is Schur-convex* $(Schur - concave)$ *on* I^2*.*

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 1. [5, 7] Let $\Omega \subseteq \mathbb{R}^n, \boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \Omega$, and let $\varphi : \Omega \to \mathbb{R}$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(2) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$. φ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only$-\varphi$ is increasing.

(3) φ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$, φ is said to be a Schur-concave function on Ω if and only $-\varphi$ is Schur-convex function.

Lemma 1 ([5, p. 5]). *Let $\boldsymbol{x} \in \mathbb{R}^n$ and $\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^n x_i$. Then*

$$(\bar{\boldsymbol{x}}, \ldots, \bar{\boldsymbol{x}}) \prec \boldsymbol{x}.$$

Lemma 2. *Let $a \le b, u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$ and $\frac{1}{2} \le t_2 \le t_1 \le 1$. Then*

$$(u(t_2), u(t_2)) \prec (u(t_1), u(t_1)) \prec (a, b) \tag{7}$$

Proof. From $a < b, \frac{1}{2} \le t_2 \le t_1 \le 1$, it is easy to see that $u(t_1) \ge v(t_1), u(t_2) \ge v(t_2), b \ge u(t_1) \ge u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = a + b$. By Definition 1, it follows that (7) holds. □

Lemma 3 ([5, p. 5]). *Let $\Omega \subseteq \mathbb{R}^n$ is symmetric and has a nonempty interior set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \ge 0 (\le 0)$$

holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

Lemma 4 ([5, p. 48–49]). *Let $\Omega \subseteq \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}, h : \mathbb{R} \to \mathbb{R}, \psi(x) = h(\varphi(x))$.*

(1) *If φ is Schur-convex and h is increasing, then ψ is also Schur-convex.*
(2) *If φ is Schur-convex and h is decreasing, then ψ is also Schur-concave.*
(3) *If φ is Schur-concave and h is increasing, then ψ is also Schur-concave.*

Lemma 5. *Let $F(\alpha, \beta) = \int_\alpha^\beta \int_\alpha^\beta f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, where $f(x, y)$ is continuous on the rectangle $[a, p; a, q]$, $\alpha = \alpha(b)$ and $\beta = \beta(b)$ are differentiable with b, $a \le \alpha(b) \le p$ and $a \le \beta(b) \le q$. Then*

$$\frac{\partial F}{\partial b} = \left(\int_\alpha^\beta f(\alpha, y)\,\mathrm{d}y\right)\alpha^{'}(b) + \left(\int_\alpha^\beta f(x, \beta)\,\mathrm{d}x\right)\beta^{'}(b) \tag{8}$$

Proof. Since $F(\alpha,\beta)=\int_\alpha^\beta\int_\alpha^\beta f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, by the derivation rule for the composite functions, we have

$$\frac{\partial F}{\partial b}=\frac{\partial F}{\partial \alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}b}+\frac{\partial F}{\partial \beta}\frac{\mathrm{d}\beta}{\mathrm{d}b}$$

which is the equation (8).

□

3. Proofs of main results

Another proof of inequalities in (2):

By the transformation $s=tx+(1-t)\frac{a+b}{2}$, we get

$$H(t)=\frac{1}{u-v}\int_v^u f(s)\,\mathrm{d}s,$$

where $u=\frac{1+t}{2}b+\frac{1-t}{2}a$ and $v=\frac{1+t}{2}a+\frac{1-t}{2}b$. From Lemma 2 and Lemma 1, we have $(u,v)\prec(a,b)$ and $\left(\frac{a+b}{2},\frac{a+b}{2}\right)\prec(u,v)$ respectively. And then by Theorem D, it follows that

$$\Phi\left(\frac{a+b}{2},\frac{a+b}{2}\right)\le\Phi(u,v)\le\Phi(a,b),$$

i.e. (2)holds.

Proof of Theorem 1:

It is sufficient prove that (i), the proof of (ii) is similar with (i). We need only consider the case of $\frac{1}{2}\le t<1$. It is clear that $P(a,b)$ is symmetric. When $a\neq b$, let

$$P_1(a,b)=\int_a^b f\left(ta+(1-t)x\right)\mathrm{d}x$$

and

$$P_2(a,b)=\int_a^b f\left(tb+(1-t)x\right)\mathrm{d}x.$$

Then

$$P(a,b)=\frac{1}{2(b-a)}\left[P_1(a,b)+P_2(a,b)\right]=G(t),a\neq b.$$

By the transformation $s=ta+(1-t)x$, we get

$$\begin{aligned}P_1(a,b)&=\frac{1}{1-t}\int_a^{ta+(1-t)b}f(s)\,\mathrm{d}s\\&=\frac{1}{1-t}\left[\int_0^{ta+(1-t)b}f(s)\,\mathrm{d}s-\int_0^a f(s)\,\mathrm{d}s\right].\end{aligned}$$

$$\frac{\partial P_1(a,b)}{\partial a} = \frac{1}{1-t}\left[f\left(ta+(1-t)b\right)t - f(a)\right]$$
$$= \frac{t}{1-t}f\left(ta+(1-t)b\right) - \frac{f(a)}{1-t}. \tag{9}$$

$$\frac{\partial P_1(a,b)}{\partial b} = \frac{1}{1-t}\left[(1-t)f\left(ta+(1-t)b\right)\right] = f\left(ta+(1-t)b\right). \tag{10}$$

Notice that $P_2(a,b) = -P_1(b,a)$, from (10), we get

$$\frac{\partial P_2(a,b)}{\partial a} = -\frac{\partial P_1(b,a)}{\partial a} = -f\left(tb+(1-t)a\right), \tag{11}$$

and from (9), we get

$$\frac{\partial P}{\partial b} = -\frac{\partial P_1(b,a)}{\partial b} = \frac{f(b)}{1-t} + \frac{t}{1-t}f\left(tb+(1-t)a\right). \tag{12}$$

And then

$$\frac{\partial P_1(a,b)}{\partial b} - \frac{\partial P_1(a,b)}{\partial a} = f\left(ta+(1-t)b\right) - \frac{t}{1-t}f\left(ta+(1-t)b\right) - \frac{f(a)}{1-t}$$
$$= \frac{1-2t}{1-t}f\left(ta+(1-t)b\right) + \frac{f(a)}{1-t},$$

$$\frac{\partial P_2(a,b)}{\partial b} - \frac{\partial P_2(a,b)}{\partial a} = \frac{f(b)}{1-t} - \frac{t}{1-t}f\left(tb+(1-t)a\right) + f\left(tb+(1-t)a\right)$$
$$= \frac{f(b)}{1-t} + \frac{1-2t}{1-t}f\left(tb+(1-t)a\right).$$

Since

$$\frac{\partial P(a,b)}{\partial b} = \left\{-\frac{1}{2(b-a)^2}[P_1(a,b)+P_2(a,b)] + \frac{1}{2(b-a)}\left[\frac{\partial P_1(a,b)}{\partial b} + \frac{\partial P_2(a,b)}{\partial b}\right]\right\}$$

and

$$\frac{\partial P(a,b)}{\partial a} = \left\{\frac{1}{2(b-a)^2}[P_1(a,b)+P_2(a,b)] + \frac{1}{2(b-a)}\left[\frac{\partial P_1(a,b)}{\partial a} + \frac{\partial P_2(a,b)}{\partial a}\right]\right\},$$

then

$$
\begin{aligned}
&(b-a)\left(\frac{\partial P(a,b)}{\partial b}-\frac{\partial P(a,b)}{\partial a}\right)\\
&=\frac{1}{2}\left[\left(\frac{\partial P_1(a,b)}{\partial b}-\frac{\partial P_1(a,b)}{\partial a}\right)+\left(\frac{\partial P_2(a,b)}{\partial b}-\frac{\partial P_2(a,b)}{\partial a}\right)\right]-G(t)\\
&=\frac{1}{2(1-t)}\left[f(a)+f(b)+(1-2t)\left(f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right)\right)\right]-G(t)\\
&\geq\frac{1}{2(1-t)}\left[f(a)+f(b)+(1-2t)\left(tf(a)+(1-t)f(b)+tf(b)+(1-t)f(a)\right)\right]-G(t)\\
&\quad\text{(notice that } f \text{ is convex and from } \frac{1}{2}\leq t<1\text{, we have } \frac{1}{2(1-t)}\leq 0)\\
&=f(a)+f(b)-G(t)\geq 0. \quad (\text{ by the right inequality in (6)})
\end{aligned}
$$

According to Lemma 3, it follows that $P(a,b)$ is Schur-convex on I^2. The proof of Theorem 1 is completed.

Proof of Theorem 2:

Taking $\alpha=\beta=b$ in Lemma 5, when $a\neq b$, we have

$$
\begin{aligned}
\frac{\partial Q(a,b)}{\partial b}=&\frac{-2}{(b-a)^3}\int_a^b\int_a^b f\left(tx+(1-t)y\right)\mathrm{d}x\,\mathrm{d}y\\
&+\frac{1}{(b-a)^2}\left[\int_a^b f\left(tb+(1-t)y\right)\mathrm{d}y+\int_a^b f\left(tx+(1-t)b\right)\mathrm{d}x\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q(a,b)}{\partial a}=&\frac{2}{(b-a)^3}\int_a^b\int_a^b f\left(tx+(1-t)y\right)\mathrm{d}x\,\mathrm{d}y\\
&+\frac{1}{(b-a)^2}\left[\int_a^b f\left(ta+(1-t)y\right)\mathrm{d}y+\int_a^b f\left(tx+(1-t)a\right)\mathrm{d}x\right]
\end{aligned}
$$

Now we only consider the case of convexity, the case of concavity is similar.

$$(b-a)\left(\frac{\partial Q(a,b)}{\partial b}-\frac{\partial Q(a,b)}{\partial a}\right)$$

$$=\frac{1}{b-a}\left[\int_a^b (f(tb+(1-t)y)+f(ta+(1-t)y))\,\mathrm{d}y+\int_a^b (f(tx+(1-t)b)+f(tx+(1-t)a))\,\mathrm{d}x\right]$$

$$-\frac{4}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{1}{b-a}\left[\int_a^b (f(tb+(1-t)x)+f(ta+(1-t)x))\,\mathrm{d}x+\int_a^b (f(tx+(1-t)b)+f(tx+(1-t)a))\,\mathrm{d}x\right]$$

$$-\frac{4}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq\frac{1}{b-a}\left[\int_a^b (f(tb+(1-t)x)+f(tx+(1-t)b))+f(ta+(1-t)x)+f(tx+(1-t)a))\,\mathrm{d}x\right]$$

$$-\frac{4}{b-a}\int_a^b f(x)\,\mathrm{d}x\quad\text{(because } f \text{ is convex)}$$

$$\geq\frac{1}{b-a}\left[\int_a^b (tf(b)+(1-t)f(x)+tf(x)+(1-t)f(b)+tf(a)+(1-t)f(x)+tf(x)+(1-t)f(a))\,\mathrm{d}x\right]$$

$$-\frac{4}{b-a}\int_a^b f(x)\,\mathrm{d}x\quad\text{(because } f \text{ is convex)}$$

$$=\frac{1}{b-a}\left[\int_a^b ((f(b)+f(x)+f(x)+f(a))\,\mathrm{d}x\right]-\frac{4}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

$$=\frac{1}{b-a}\left[\int_a^b ((f(b)+f(a)-2f(x))\,\mathrm{d}x\right]$$

$$=f(b)+f(a)-\frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x\geq 0.$$

According to Lemma 3, it follows that $Q(a,b)$ is Schur-convex on I^2. The proof of Theorem 2 is completed.

4. Applications

Theorem 3. *Let* $t\in[0,1)\,,a,b\in\mathbb{R}_+=[0,+\infty)$*, and let*

$$L_r(a,b;t)=\left[\frac{(b^r-a^r)-(u^r-v^r)}{r(1-t)(b-a)}\right]^{\frac{1}{r-1}}\,,a\neq b$$

$$L_r(a,a;t)=a$$

where $u=tb+(1-t)a, v=ta+(1-t)b$.

(i) if $r > 2$ *and* $\frac{1}{2} \leq t \leq 1$*, then* $L_r(a,b;t)$ *is Schur-convex on* $\mathbb{R}_+^2$*,*

(ii) if $1 < r < 2$ *and* $0 \leq t \leq \frac{1}{2}$*, then* $L_r(a,b;t)$ *is Schur-concave on* $\mathbb{R}_+^2$*,*

(iii) if $r \leq 1, r \neq 0$ *and* $\frac{1}{2} \leq t \leq 1$*, then* $L_r(a,b;t)$ *is Schur-concave on* $\mathbb{R}_+^2$*.*

Proof. Taking $f(x) = x^{r-1}, r \neq 0$, then for $a \neq b$, from Theorem 1, we have

$$\begin{aligned} G(t) &= \frac{1}{2(b-a)} \int_a^b \left[(ta + (1-t)x)^{r-1} + (tb + (1-t)x)^{r-1} \right] \mathrm{d}x \\ &= \frac{1}{2r(1-t)(b-a)} \left[(ta + (1-t)x)^r \Big|_a^b + (tb + (1-t)x)^r \Big|_a^b \right] \\ &= \frac{(b^r - a^r) + (ta + (1-t)b)^r - (tb + (1-t)a)^r}{2r(1-t)(b-a)} \\ &= \frac{(b^r - a^r) + (u^r - v^r)}{2r(1-t)(b-a)}. \end{aligned}$$

(i) if $r > 2$ and $\frac{1}{2} \leq t \leq 1$, since $f(x) = x^{r-1}$ is convex on $\mathbb{R}_+$, from Theorem 1 we obtain that $P(a,b)$ is Schur-convex on $\mathbb{R}_+^2$. Furthermore, since $h : t \to t^{\frac{1}{r-1}}$ is increasing on $\mathbb{R}_+$, then from (1) in Lemma 4, $L_r(a,b;t) = [P(a,b)]^{\frac{1}{r-1}}$ is Schur-convex on $\mathbb{R}_+^2$,

(ii) if $1 < r < 2$ and $0 \leq t \leq \frac{1}{2}$, since $f(x) = x^{r-1}$ is concave on $\mathbb{R}_+$, from Theorem 1 we obtain that $P(a,b)$ is Schur-concave on $\mathbb{R}_+^2$. Furthermore, since $h : t \to t^{\frac{1}{r-1}}$ is increasing on $\mathbb{R}_+$, then from (3) in Lemma 4, $L_r(a,b;t) = [P(a,b)]^{\frac{1}{r-1}}$ is Schur-concave on $\mathbb{R}_+^2$,

(iii) if $r \leq 1, r \neq 0$ and $\frac{1}{2} \leq t \leq 1$, since $f(x) = x^{r-1}$ is convex on $\mathbb{R}_+$, from Theorem 1 we obtain that $P(a,b)$ is Schur-convex on $\mathbb{R}_+^2$. Furthermore, since $h : t \to t^{\frac{1}{r-1}}$ is decreasing on $\mathbb{R}_+$, then from (2) in Lemma 4, $L_r(a,b;t)$ is Schur-concave on $\mathbb{R}_+^2$. Setting $r \to 1$, it is deduced that $L_r(a,b;t)$ is still Schur-concave on $\mathbb{R}_+^2$ for $r = 1$. The proof of Theorem 3 is completed. □

Corollary 1. *If* $(r,t) \in \{r > 2, \frac{1}{2} \leq t \leq 1\}$*, then*

$$\frac{a+b}{2} \leq L_r(a,b;t) \leq (a+b) \left(\frac{(t^r - 1)^r + (1-t)^r}{2r(t-1)} \right)^{\frac{1}{r-1}} \tag{13}$$

if $(r,t) \in \{1 < r < 2, 0 \leq t \leq 1/2\} \cup \{r \leq 1, r \neq 0, 1/2 \leq t \leq 1\}$*, then the two inequalities in* (9) *are all reversed.*

Proof. Since

$$\left(\frac{a+b}{2}, \frac{a+b}{2} \right) \prec (a,b) \prec (a+b, 0),$$

then from Theorem 3, when $r > 2$ and $\frac{1}{2} \le t \le 1$, we have

$$L_r\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \le L_r(a,b) \le L_r(a+b, 0),$$

i.e. (13) is holds. When $(r,t) \in \{1 < r < 2, 0 \le t \le 1/2\} \cup \{r \le 1, r \ne 0, 1/2 \le t \le 1\}$, the two inequalities in (13) are all reversed. □

Remark 2. $L_r(a,b;0)$ is the generalized logarithmic mean (or Stolarsky's mean):

$$S_r(a,b) = \left(\frac{b^r - a^r}{r(b-a)}\right)^{\frac{1}{r-1}}.$$

Taking $t = 0$, from (13), we can obtain known inequality [5, p. 53]:

$$\frac{a+b}{2} \le S_r(a,b) \le \frac{a+b}{r^{\frac{1}{r-1}}},\ r > 2 \tag{14}$$

Theorem 4. *Let* $\frac{1}{2} \le t < 1, a, b \in \mathbb{R}_+$, *and let*

$$L(a,b;t) = \frac{(\ln b - \ln a) - (\ln u - \ln v)}{2(1-t)(b-a)}, a \ne b$$

$$L(a,a;t) = a^{-1}$$

where $u = tb + (1-t)a, v = ta + (1-t)b$. *Then* $L(a,b;t)$ *is Schur-convex on* $\mathbb{R}_+^2$.

Proof. Taking $f(x) = x^{-1}$, then for $a \ne b$, from Theorem 1, we have

$$\begin{aligned} G(t) &= \frac{1}{2(b-a)} \int_a^b \left[(ta + (1-t)x)^{-1} + (tb + (1-t)x)^{-1}\right] \mathrm{d}x \\ &= \frac{1}{2(1-t)(b-a)} \left[\ln(ta + (1-t)x)\Big|_a^b + \ln(tb + (1-t)x)\Big|_a^b\right] \\ &= \frac{(\ln b - \ln a) - [\ln(tb + (1-t)a) - \ln(ta + (1-t)b)]}{2(1-t)(b-a)} \\ &= \frac{(\ln b - \ln a) - (\ln u - \ln v)}{2(1-t)(b-a)} \end{aligned}$$

Since $f(x) = x^{-1}$ is convex on $\mathbb{R}_+$, when $\frac{1}{2} \le t < 1$, by Theorem 1, it follows that $L(a,b;t)$ is Schur-convex on $\mathbb{R}_+^2$. □

From Theorem 4 and $\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (a,b)$, we get

Corollary 2. *Let* $\frac{1}{2} \le t < 1, a, b \in \mathbb{R}_+$. *Then*

$$L(a,b;t) \le \frac{2}{a+b}. \tag{15}$$

Remark 3. $L(a,b;0)$ is the logarithmic mean $L(a,b)=\frac{\ln b-\ln a}{b-a}$. Taking $t=0$, from (15), we can obtain the Ostle-Terwilliger inequality [7]:

$$\frac{\ln b-\ln a}{b-a}\le\frac{2}{a+b}. \tag{16}$$

Theorem 5. *Let $t\in(0,1)$, $a,b\in\mathbb{R}_+=[0,+\infty)$, and let*

$$Q_r(a,b;t)=\left[\frac{a^{r+1}+b^{r+1}-\left(u^{r+1}-v^{r+1}\right)}{r(r+1)t(1-t)(b-a)^2}\right]^{\frac{1}{r-1}},a\ne b$$

$$Q_r(a,a;t)=a$$

where $u=tb+(1-t)a, v=ta+(1-t)b$, if $r\ge 2$, then $Q_r(a,b;t)$ is Schur-convex on $\mathbb{R}_+^2$, if $r\in\{1\le r<2\}\cup\{r<1,r\ne 0,-1\}$, then $Q_r(a,b;t)$ is Schur-concave on $\mathbb{R}_+^2$.

Proof. Taking $f(x)=x^{r-1}, r\ne 0$, then for $a\ne b$,

$$\begin{aligned}
Q(a,b)=F(t)&=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(tx+(1-t)y\right)\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{rt(b-a)^2}\int_a^b\left(tx+(1-t)y\right)^r\Big|_a^b\,\mathrm{d}y\\
&=\frac{1}{rt(b-a)^2}\int_a^b\left[\left(tb+(1-t)y\right)^r-\left(ta+(1-t)y\right)^r\right]\mathrm{d}y\\
&=\frac{1}{r(r+1)t(1-t)(b-a)^2}\left[\left(tb+(1-t)y\right)^{r+1}-\left(ta+(1-t)y\right)^{r+1}\right]_a^b\\
&=\frac{b^{r+1}-\left(ta+(1-t)y\right)^{r+1}-\left(tb+(1-t)y\right)^{r+1}+a^{r+1}}{r(r+1)t(1-t)(b-a)^2}\\
&=\frac{a^{r+1}+b^{r+1}-\left(u^{r+1}-v^{r+1}\right)}{r(r+1)t(1-t)(b-a)^2}.
\end{aligned}$$

The following discussions are similar with Theorem 3, hence it is omitted. Theorem 5 is completed. □

Corollary 3. *When $r\ge 2$, we have*

$$\frac{a+b}{2}\le Q_r(a,b;t)\le(a+b)\left(\frac{1-(1-t)^{r+1}-r^{r+1}}{r(r+1)t(1-t)}\right)^{\frac{1}{r-1}} \tag{17}$$

when $r\in\{1\le r<2\}\cup\{r<1,r\ne 0,-1\}$, the two inequalities in (17) are all reversed.

Proof. Since

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (a,b) \prec (a+b,0),$$

then from Theorem 5, when $r \geq 2$ we have

$$Q_r\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \leq Q_r(a,b) \leq Q_r(a+b,0),$$

i.e. (17) is holds. when $r \in \{1 \leq r < 2\} \cup \{r < 1, r \neq 0, -1\}$, the two inequalities in (17) are all reversed. □

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