ON AN INEQUALITY OF ALZER FOR NEGATIVE POWERS

JÓZSEF SÁNDOR

Abstract. By using a method of [5] we obtain a simpler proof of a result of Chao-Ping Chen and Feng Qi [2]. By a method of Kuang Jichang [4], an extension is provided.

1. Introduction

In 1993 H. Alzer [1] discovered the following result: For all positive integers $n \geq 1$ and all real numbers $r \geq 0$ one has

$$\left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r} \geq \frac{n}{n+1} \tag{1}$$

In [5] we have offered a simple proof of (1) based on mathematical induction and Cauchy's mean value theorem of differential calculus. We note that for $r > 0$ there is strict inequality in (1). Recently, Chao-Ping Chen and Feng Qi [2] discovered the interesting fact that (1) holds true also for $r < 0$. Their method is based on mathematical induction and a complicated function combined with Jensen's inequality.

Our aim in what follows is to show that the method of [5] can be applied here, again. On the other hand, we shall obtain an extension based on a method from [4].

2000 Mathematics Subject Classification: 26D15

Key words and phrases: convex function, Alzer's inequality.

2. New proof

First remark that the mathematical induction process (see [2]) leads to the following inequality:

$$(k+1)^s[(k+1)^{1-s}-k^{1-s}] > (k+2)^s[(k+2)^{1-s}-(k+1)^{1-s}], \qquad (2)$$

where $0 < s < 1$ and $k \geq 1$ is a positive integer.

In order to prove (2), let the functions $f, g : [k, k+1] \to \mathbb{R}$ be given by

$$f(x) = (x+1)^{1-s}, \quad g(x) = x^{1-s} \quad (k \leq x \leq k+1)$$

By the Cauchy mean value theorem one can write:

$$\frac{f(k+1)-f(k)}{g(k+1)-g(k)} = \frac{f'(\xi)}{g'(\xi)}, \text{ with } \xi \in (k, k+1).$$

Since

$$\frac{f'(\xi)}{g'(\xi)} = \left(\frac{\xi}{\xi+1}\right)^s < \left(\frac{k+1}{k+2}\right)^s,$$

we immediately get

$$\frac{(k+2)^{1-s}-(k+1)^{1-s}}{(k+1)^{1-s}-k^{1-s}} < \frac{(k+1)^s}{(k+2)^s},$$

which implies relation (2).

Remark. This method appears also in the author's paper [6].

3. An extension

Inequality (1) for $r = -s$ ($s > 0$) is in fact equivalent to

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{n+1}{i}\right)^s > \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n}{i}\right)^s \qquad (3)$$

In what follows, we shall obtain the following more general result:

Theorem. *Suppose that $f : (0,1] \to \mathbb{R}$ is a strictly decreasing, convex function. Then one has the inequality*

$$\frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) > \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right). \tag{4}$$

Proof. The proof is based on the method of J. Kuang [4], who showed that (4) holds true with reversed sign inequality, when f is strictly increasing and concave or convex function. Since f is convex, one can write the relations

$$\begin{aligned}&\frac{i}{n+1} f\left(\frac{i+1}{n+1}\right) + \left(1 - \frac{i}{n+1}\right) f\left(\frac{i}{n+1}\right)\\ &\geq f\left[\frac{i(i+1)}{(n+1)^2} + \left(1 - \frac{i}{n+1}\right)\frac{i}{n+1}\right]\end{aligned} \tag{5}$$

for $i = 1, 2, \ldots, n$.

Since

$$\frac{i(i+1)}{(n+1)^2} + \left(1 - \frac{i}{n+1}\right)\frac{i}{n+1} = \frac{i(n+2)}{(n+1)^2} \text{ and } n(n+2) < (n+1)^2,$$

and f being strictly decreasing, the right side of (5) is greater than

$$f\left(\frac{i(n+1)^2}{n(n+1)^2}\right) = f\left(\frac{i}{n}\right).$$

Therefore, one has

$$\frac{i}{n+1} f\left(\frac{i+1}{n+1}\right) + \left(1 - \frac{i}{n+1}\right) f\left(\frac{i}{n+1}\right) > f\left(\frac{i}{n}\right) \tag{6}$$

By summation we get

$$\sum_{i=1}^{n}\left[\frac{i}{n+1} f\left(\frac{i+1}{n+1}\right) + \left(1 - \frac{i}{n+1}\right) f\left(\frac{i}{n+1}\right)\right] > \sum_{i=1}^{n} f\left(\frac{i}{n}\right),$$

i.e.

$$\sum_{i=1}^{n} \frac{i}{n+1} f\left(\frac{i+1}{n+1}\right) + \sum_{i=1}^{n} \left(\frac{n}{n+1} - \frac{i-1}{n+1}\right) f\left(\frac{i}{n+1}\right)$$

$$= \frac{n}{n+1} f(1) + \frac{n}{n+1} \sum_{i=1}^{n} f\left(\frac{i}{n+1}\right)$$

$$= \frac{n}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) > \sum_{i=1}^{n} f(i),$$

giving relation (4).

By letting $f(x) = \dfrac{1}{x^s} = x^{-s}$, which is convex and strictly decreasing, (4) is applicable, so relation (3) is reobtained.

Remark. Inequality (4) holds true also when f is concave and strictly decreasing. This follows by

$$\frac{i-1}{n} f\left(\frac{i-1}{n}\right) + \left(1 - \frac{i-1}{n}\right) f\left(\frac{i}{n}\right) \le f\left[\left(\frac{i-1}{n}\right)^2 + \left(1 - \frac{i-1}{n}\right) \frac{i}{n}\right]$$

$$= f\left(\frac{ni-i+1}{n^2}\right) < f\left(\frac{i}{n+1}\right),$$

since $(n+1)(ni-i+1) > n^2 i$ (i.e. $n+1 > i$). After summation for $i = 1, 2, \ldots, n$, relation (4) follows.

FINAL NOTES. After this paper was completed, the author discovered that, the inequality of Alzer for negative powers was proved (even in an improved form) by Alzer himself, in the following paper:

H. Alzer, Refinement of an inequality of G. Bennett, *Discrete Mathematics*, **135** (1994), 39–46.

Our result offers a new proof, and an extension for convex functions.

References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179**(1993), 396-402.

[2] C.-P. Chen and F. Qi, *Extension of an inequality of Alzer for negative powers*, Tamkang J. Math. **36**(2004), no.1, 69-72.

[3] J.-Ch. Kuang, *Some extensions and refinements of Minc-Sathre inequality*, Math. Gaz. **83**(1999), 123-127.

[4] J. Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. **192**(1995), 1034-1035.

[5] J. Sándor, *On an inequality of Alzer, II*, Octogon Math. Mag. **11**(2003), no.2, 554-555.

Department of Mathematics
and Computer Sciences
Babeş-Bolyai University of Cluj
ROMANIA
e-mail addresses: jjsandor@hotmail.com,
jsandor@math.ubbcluj.ro