# An extension of $k$-perfect numbers

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ABSTRACT. We extend the definition of $k$-perfect numbers for certain special rational numbers $k$. A characterization of a class of such perfect numbers is given. A theorem on 3/2-cvasi perfect numbers is proved, too.

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## 1 Introduction

Let $\sigma(n)$ denote the sum of all distinct positive divisors of $n$. It is well-known that $n$ is called **perfect**, if $\sigma(n) = 2n$. The even perfect numbers are characterized by the famous Euclid-Euler theorem:

**Theorem 1.** *The number $n$ is an even perfect number if and only if $n$ has the form $n = 2^a(2^{a+1} - 1)$, where $a \geq 1$ is an integer, and $2^{a+1} - 1$ is a prime ("Mersenne prime").*

For the history of this theorem, new proofs and various related themes see Chapter 1 of our book [3].

The number $n$ is called **$k$-perfect** if

$$\sigma(n) = kn. \tag{1}$$

This notion is well-known, when $k$ is a positive integer. However, we may assume that, $k$ is a rational number. For certain special rational numbers, an extension of Theorem 1 is obtainable. Particularly, when $k = \frac{3}{2}$, the number $n$ will be called a **$\frac{3}{2}$-perfect** number. It is well-known that, a number $n$ is called **cvasi-perfect**, if

$$\sigma(n) = 2n + 1 \tag{2}$$

It is not known if there exist such numbers (see e.g. [2], [3]). Similarly, $n$ is **pseudo-perfect**, if $\sigma(n) = 2n - 1$. Then $n = 2^a$ ($a \geq 1$ integer) are pseudo-perfect numbers. It is not known if there exist such numbers which are not powers of 2.

In what follows, $n$ will be called **$\frac{3}{2}$-cvasi-perfect**, if

$$\sigma(n) = \frac{3}{2}n + 1 \tag{3}$$

We shall prove that $n = 4$ is the single number with this property.

## 2 Main results

Let $p \geq 2$ be a prime number, and define $k = k_p = \dfrac{p}{p-1}$.

We say that $n$ is a $k_p$-**perfect** number, if (1) holds with $k = k_p$. The following extension of Theorem 1 holds true:

**Theorem 2.** *All $k_p$-perfect numbers divisible by $p$ have the form $n = p^a(p^{a+1}-1)$, where $a \geq 1$ is an integer, and $p^{a+1}-1$ is a prime.*

**Remark.** For $p = 2$, Theorem 1 is reobtained, as $k_2 = 2$.

For the proof of Theorem 2 we need the following auxiliary result:

**Lemma 1.** *1) One has $\sigma(ab) \geq a\sigma(b)$ for all $a, b \geq 1$; with equality only for $a = 1$;*

*2) $\sigma(a) \geq a + 1$ for $a \geq 2$, with equality only for $a =$ prime.*

**Proof.** 2) is trivial since for $a \geq 2$, 1 and $a$ are distinct divisors of $a$, and only for $a =$ prime there is equality.

For the proof of 1), let $d|b$ be an arbitrary divisor of $b$. Then $a \cdot d$ is a divisor of $a \cdot b$. Writing this for all divisors of $b$, after summation we get the desired inequality. If $a > 1$, there are also other divisors, e.g. $1 \neq a \cdot d$.

**Proof of Theorem 2.** Let $p|n$; then $n = p^a N$, where $(p, N) = 1$. By the multiplicativity of $\sigma$ function, from (1) with $k = k_p$ one gets

$$\sigma(n) = \frac{p^{a+1}-1}{p-1}\sigma(N) = \frac{p}{p-1}n,$$

so

$$(p^{a+1}-1)\sigma(N) = p^{a+1}N \tag{4}$$

By $(p^{a+1}, p^{a+1}-1) = 1$, $N$ must be a multiple of $p^{a+1}-1$, i.e. $N =$

$(p^{a+1}-1)M$. From (4) we get

$$\sigma[(p^{a+1}-1)M] = p^{a+1}M \tag{5}$$

Now by the Lemma,

$$\sigma[(p^{a+1}-1)M] \geq M\sigma(p^{a+1}-1) \geq p^{a+1}M,$$

so $p^{a+1}M \geq M\sigma(p^{a+1}-1) \geq p^{a+1}M$, which shows that there is equality in all inequalities. This may happen only when $M = 1$ and $p^{a+1}-1$ is prime, i.e. when $N = p^{a+1}-1$ and $n = p^a(p^{a+1}-1)$.

**Example.** Let $p = 3$. Then $k_p = \dfrac{3}{2}$, and

$$\sigma(n) = \frac{3}{2}n \tag{6}$$

By Theorem (2), all solutions of (6) which are multiples of 3 have the general form $n = 3^a(3^{a+1}-1)$.

One may ask the following (open) question: what is the general form of numbers $n$ (which are necessary even) satisfying (6) and not divisible by 3?

**Theorem 2.** *The only $\dfrac{3}{2}$-cvasi-perfect number $n$ is $n = 4$.*

**Proof.** $n = 4$ is a solution, since $\sigma(4) = 7$ and $\dfrac{3}{2} \cdot 4 + 1 = 7$. Let us suppose now that $n$ satisfies relation (3). Since $n$ is even, let $n = 2^a N$, where $a \geq 1$ and $N$ odd. By $\sigma(2^a) = 2^{a+1} - 2 = 4 \cdot 2^{a-1} - 1$, (3) may be equivalently written also as

$$(4 \cdot 2^{a-1} - 1)\sigma(N) = 3 \cdot 2^{a-1}N + 1 \tag{7}$$

Let $\sigma(N) = N + T$, where $T \geq 0$. By (7) we get

$$(4 \cdot 2^{a-1} - 1)(N + T) - 3 \cdot 2^{a-1} N = 1,$$

i.e.

$$N(2^{a-1} - 1) + T(2^{a+1} - 1) = 1 \tag{8}$$

Since $2^{a-1} - 1 \geq 0$ for all $a \geq 1$ and $2^{a_1} - 1 > 1$, (8) is possible only when $2^{a-1} - 1 = 1$ and $T = 0$, i.e. when $a = 2$ and $N = 1$. This gives $n = 2^2 \cdot 1 = 4$.

**Remarks.** 1) It is not known if there exist odd perfect numbers. What are the $k_p$-perfect numbers, which are not multiples of $p$ (see Theorem 2)?

2) A. Makowski (see [3]) has studied the equation

$$\sigma(n) = 2n + 2 \tag{9}$$

As particular solutions, we get $n = 2^a(2^{a+1} - 3)$, where $2^{a+1} - 3$ are primes. It is not known it there are infinitely many such primes. Are there other solutions to (9)?

3) Equations of type (9) for rational coefficients are e.g. the following:

$$\sigma(n) = \frac{3}{2}n + \frac{1}{2} \tag{10}$$

or

$$\sigma(n) = \frac{3}{2}n + \frac{3}{2} \tag{11}$$

It is immediately seen that if $n = 3^{a+1} - 2$ are primes, then $n = 3^a(3^{a+1} - 2)$ are solution to (10). Similarly, $n = 3^a(3^{a+1} - 4)$ for (11), when $3^{a+1} - 4$ are primes. A computer search for such primes has been done by A. Bege and K. Fogarasi [1]. It is not known the general form of solutions to (10) or (11).

## References


[1] A. Bege and K. Fogarasi, *Personal communication* (november 2005).

[2] R.K. Guy, *Unsolved problems in number theory*, 2nd ed., Springer Verlag, 2004.

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