SHARP INEQUALITIES INVOLVING THE CONSTANT e AND THE SEQUENCE $(1+1/n)^n$

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Abstract. Several new and sharp inequalities involving the constant e and the sequence $(1+1/n)^n$ are proved.

1. INTRODUCTION

The constant e (or Euler's number) which is usually represented by

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad or \quad e=\sum\nolimits_{n=0}^{\infty}\frac{1}{n!}$$

is one of the most important constants in mathematics and science. In addition to many other applications, it is involved in some mathematical inequalities. For example, the well-known Carleman's inequality[3]

$$\sum_{n=1}^{\infty}(a_1a_2...a_n)^{1/n}<e\sum_{n=1}^{\infty}a_n,$$

where $a_n\geq 0$ $(n=1,2,...)$ and $\sum_{n=1}^{\infty}a_n<\infty$, and its Polya's generalization[2]

$$\sum_{n=1}^{\infty}\lambda_n\left(a_1^{\lambda_1}a_2^{\lambda_2}...a_n^{\lambda_n}\right)^{1/\sigma_n}<e\sum_{n=1}^{\infty}\lambda_na_n,$$

where $\lambda_n>0,\ \sigma_n=\sum_{k=1}^{n}\lambda_k,\ a_n\geq 0(n=1,2,...)$ and $0<\sum_{n=1}^{\infty}\lambda_na_n<\infty$, are nice examples of applications of approximation of e.

Recently, some authors have obtained interesting inequalities for this important constant and applied them to refine Hardy and Carleman's inequalities. We list here some of them: Zitian, and Yibing[19]

$$e\left(1-\frac{7}{14x+12}\right)<\left(1+\frac{1}{x}\right)^x<e\left(1-\frac{6}{12x+11}\right)\ (x\geq 1),$$

Sandor[9]

$$e\left(1+\frac{1}{x}\right)^{-1/2}\exp\left(\frac{1}{3(2x+1)^2}\right)<\left(1+\frac{1}{x}\right)^x$$
$$<e\left(\frac{2x+1}{2x+2}\right)\exp\left(-\frac{1}{6(2x+1)^2}\right)$$

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and Yang[15, Lemma1]

$$\left(1+\frac{1}{x}\right)^x < e\left[1-\frac{1}{2(1+x)}-\frac{1}{24(1+x)^2}-\frac{1}{48(1+x)^3}\right].$$

Also, in [10] and [11] Sandor proved the following nice inequalities:

$$\left(1+\frac{1}{n}\right)^{n+\alpha} < e < \left(1+\frac{1}{n}\right)^{n+\beta},$$

where $\alpha = \frac{1}{\log 2} - 1 = 0.44269...$and $\beta = 0.5$ are best possible constants, and

$$1+\frac{a}{n} < \frac{e}{(1+1/n)^n} < 1+\frac{b}{n}, \tag{1.1}$$

with best possible constants $a = e/2 - 1$ and $b = 1/2$, respectively. Some other similar results can be found in ([4, 5, 12, 14, 16, 17, 18]). In this note we aim to establish some more new and sharp inequalities for this important constant and the sequence $(1 + 1/n)^n$. Since some inequalities we used in the proofs of our main results are simple consequences of inequalities for some special means, we want to recall them here briefly. The arithmetic, geometric, logarithmic and identric means of the positive real numbers a and b are defined as: $A = A(a, b) = (a + b)/2$, $G = G(a, b) = \sqrt{ab}$, $L = L(a, b) = (b - a)/(\log b - \log a)$ and $I = I(a, b) = (1/e)(b^b/a^a)^{1/(b-a)}$, respectively. These means satisfy the following inequalities:

$$G \le L, \tag{1.2}$$

which is due to Carlson[1], and

$$L \le I, \tag{1.3}$$

which is due to Stolarsky, see[13]. For many other interesting inequalities for these important means refer to ([6, 7, 8]).

2. MAIN RESULTS

Now we are in a position to establish our main results.

Theorem 2.1. *For all positive integers $n \ge 1$, we have*

$$\frac{e}{(n+1)\log(1+1/n)} < \left(1+\frac{1}{n}\right)^n \le \frac{4\log 2}{(n+1)\log(1+1/n)}, \tag{2.1}$$

where the constants $e = 2.71828...$ *and* $4log2 = 2.77259...$ *are best possible.*

Proof. Define for $x > 0$

$$\eta(x) = (x+1)^{x+1}x^{-x}\log(1+1/x). \tag{2.2}$$

Differentiation yields

$$\eta'(x) = (x+1)^{x+1}x^{-x}\left(\log^2\left(1+\frac{1}{x}\right) - \frac{1}{x^2+x}\right).$$

Since

$$L(x, x+1) = \frac{1}{\log(1+1/x)},$$

and

$$G(x, x+1) = \sqrt{x^2+x}\ ,$$

by (1.2) we have

$$\log^2(1+1/x) < \frac{1}{x^2+x}. \tag{2.3}$$

So, we find that $\eta'(x) < 0$ for $x > 0$. We can easily show that $\lim_{x\to\infty}\eta(x) = e$, concluding for $n = 1, 2, 3, \ldots$

$$e = \eta(\infty) < \eta(n) \leq \eta(1) = 4\log 2. \tag{2.4}$$

Replacing the value of η here proves Theorem 2.1. Since $e.I(x, x+1) = (x+1)(1+1/x)^x$, where I is identric mean, we want to note that the left hand side of (2.1) can be derived by using the mean inequality $L(x, x+1) \leq I(x, x+1)$ given in (1.3). □

Theorem 2.2. *For all positive integers n the following inequalities hold:*

$$\frac{4}{n+1}\left(\frac{\log 2}{\log(1+1/n)}\right)^{\alpha(1)} < \left(1+\frac{1}{n}\right)^n \leq \frac{4}{n+1}\left(\frac{\log 2}{\log(1+1/n)}\right)^{\alpha(n)},$$

where $\alpha(x) = (x^2+x)\log^2(1+1/x)$.

Proof. By mean value theorem we have a ϕ such that $0 < \phi = \phi(t) < 1$ and

$$\log(1+1/t) = \frac{1}{t+\phi(t)},\ t > 0. \tag{2.5}$$

Define

$$\rho(t) = \frac{1}{e^{1/t}-1}. \tag{2.6}$$

It is not difficult to verify that

$$\phi(\rho(t)) = t - \rho(t). \tag{2.7}$$

Integrating both sides of (2.5) over $1 \leq t \leq x$, we get

$$\int_1^x \frac{dt}{t+\phi(t)} = (x+1)\log(x+1) - x\log x - 2\log 2. \tag{2.8}$$

Inducing the change of variable $t = \rho(u)$ here, where ρ is as defined by (2.6), and using (2.7) we get for $x \geq 1$

$$\int_{1/\log 2}^{x\,+\,\phi(x)} \frac{\rho'(u)\,du}{u} = (x+1)\log(x+1) - x\log x - 2\log 2\,. \tag{2.9}$$

Differentiation of (2.6) twice yields

$$(e^u-1)^3\rho''(1/u) = u^3e^u\left(ue^u - 2e^u + u + 2\right) = u^3e^u \sum\nolimits_{k=3}^{\infty} \frac{(k-2)}{k!}u^k > 0.$$

Hence, ρ' is strictly increasing on $(\,0\,,\infty)$. Since $\rho'(u)$ is positive for $u > 0$, we obtain from (2.9) that

$$\rho'\left(1/\log 2\right)\left(\log\log 2 + \log\left(x+\phi(x)\right)\right) < (x+1)\log(x+1) - x\log x - 2\log 2$$
$$< \rho'\left(x+\phi\left(x\right)\right)\left(\log\log 2 + \log\,\left(x+\phi\left(x\right)\right)\right).$$

Since

$$x+\phi(x) = \frac{1}{\log(1+1/x)},$$

simplifying these inequalities we get for $x \geq 1$

$$\left(\frac{\log 2}{\log(1+1/x)}\right)^{2\log^2 2} < \frac{x+1}{4}\left(1+\frac{1}{x}\right)^x < \left(\frac{\log 2}{\log(1+1/x)}\right)^{(x^2+x)\log^2(1+1/x)}, \tag{2.10}$$

proving Theorem 2.2. □

Corollary 2.3. *For all positive integers n the following double inequality holds:*

$$\frac{4}{n+1}\left(\frac{\log 2}{\log(1+1/n)}\right)^a \leq \left(1+\frac{1}{n}\right)^n \leq \frac{4}{n+1}\left(\frac{\log 2}{\log(1+1/n)}\right)^b, \tag{2.11}$$

where $a = 2\log^2 2 =$ 0.960906... and $b = 1$ are best possible constants.

Proof. From (2.10) we get for $x \geq 1$

$$\left(2\log^2 2\right)\log\left(\frac{\log 2}{\log(1+1/x)}\right) < \log\left(\frac{(x+1)^{x+1}}{4x^x}\right)$$
$$< (x^2+x)\log^2(1+1/x)\log\left(\frac{\log 2}{\log(1+1/x)}\right).$$

By (2.3) we get for $x \geq 1$

$$\left(2\log^2 2\right)\log\left(\frac{\log 2}{\log(1+1/x)}\right) < \log\left(\frac{(x+1)^{x+1}}{4x^x}\right) < \log\left(\frac{\log 2}{\log(1+1/x)}\right).$$

This proves that (2.11) is satisfied for $a = 2\log^2 2$ and $b = 1$. Now we assume that the right inequality of (2.11) holds. Then we have to have

$$b \geq \lim_{n\to\infty} \frac{\log\left(\left(\frac{n+1}{4}\right)\left(1+\frac{1}{n}\right)^n\right)}{\log\left(\frac{\log 2}{\log(1+1/n)}\right)} = 1.$$

Similarly, from the left inequality of (2.11) we can write

$$a \leq \lim_{n\to 1} \frac{\log\left(\left(\frac{n+1}{4}\right)\left(1+\frac{1}{n}\right)^n\right)}{\log\left(\frac{\log 2}{\log(1+1/n)}\right)} = 2\log^2 2,$$

so that the constants a and b are best possible, completing the proof of the Corollary. □

In the following we establish a companion of (1.1).

Theorem 2.4. *For all positive integers n the following inequalities hold:*

$$e\left(1+\frac{a}{n}\right) < \left(1+\frac{1}{n}\right)^{n+1} < e\left(1+\frac{b}{n}\right), \tag{2.12}$$

where $a = 4/e - 1 = 0.47151776\ldots$ and $b = 0.5$ are best possible constants.

Proof. Let

$$\varphi(x) = (x+1)^{x+1}x^{-x}e^{-1} - x,\ x > 0. \tag{2.13}$$

Differentiate (2.13) to get

$$\varphi'(x) = e^{-1}\left[(x+1)^{x+1}x^{-x}\log(1+1/x) - e\right] = e^{-1}(\eta(x) - e),$$

where η is as defined in (2.2). By (2.4) we have $\eta(x) > e$ for $x \geq 1$. By this fact, we arrive at that φ is strictly increasing on $(0,\infty)$. One can easily show that $\lim_{x\to\infty}\varphi(x) = 1/2$. Hence we have for $n = 1, 2, 3, \ldots$

$$4/e - 1 = \varphi(1) < \varphi(n) < \varphi(\infty) = 1/2,$$

from which the proof follows. □

Theorem 2.5. *Let $n \geq 1$ be an integer. Then we have*

$$\exp\left(1 - \frac{n}{2(n+c)^2}\right) < \left(1+\frac{1}{n}\right)^n < \exp\left(1 - \frac{n}{2(n+d)^2}\right), \tag{2.14}$$

where the constants

$$c = \frac{1}{\sqrt{2-2\log 2}} - 1 = 0.27649... \text{ and } d = 1/3 = 0.33333...$$

are best possible.

Proof. Define for $x > 0$

$$\omega(x) = \frac{1}{\sqrt{2\left(\frac{1}{x} - \log\left(1+\frac{1}{x}\right)\right)}} - x. \tag{2.15}$$

We shall show that ω is strictly increasing on $(0,\infty)$. In order to fulfill this it is enough to show that $\omega'(1/x) > 0$ for $x > 0$. Now we have

$$-\frac{1}{x^2}\omega'(1/x) = \frac{(2(x-\log(x+1)))^{3/2} - x^3/(x+1)}{x^2(2x - 2\log(x+1))^{3/2}}.$$

Hence, in order to show $\omega'(1/x) > 0$ we only need to show

$$(2x - 2\log(x+1))^{3/2} - \frac{x^3}{x+1} < 0,$$

or equivalently

$$v(x) := 2(x+1)^{2/3}(x - \log(x+1)) - x^2 < 0.$$

Differentiation successively we get $v(0) = v'(0) = v''(0) = 0$ and

$$v'''(x) = -\frac{4}{27}(x+1)^{-7/3}(5x + 4\log(x+1)) < 0,$$

which proves $v(x) < 0$ for $x > 0$. So, ω is strictly increasing on $(0,\infty)$. One can easily check that $\lim_{x\to\infty}\omega(x) = 1/3$, concluding for n=1, 2 ,3,. . .

$$c = \frac{1}{\sqrt{2-2\log 2}} - 1 = 0.27649... = \omega(1) \le \omega(n) < \omega(\infty) = 1/3 = d.$$

Using (2.15) and then simplifying this inequality we prove Theorem 2.5. □

Theorem 2.6. *Let $n \ge 1$ be an integer. Then the following double inequality holds*

$$\exp\left(1 + \frac{1}{2(n+\alpha)}\right) < \left(1+\frac{1}{n}\right)^{n+1} < \exp\left(1 + \frac{1}{2(n+\beta)}\right), \tag{2.16}$$

where

$$\alpha = 1/3 = 0.3333333... \text{ and } \beta = \frac{1}{4\log 2 - 2} - 1 = 0.2943497... \tag{2.17}$$

are best possible constants.

Proof. We make the following auxiliary function for $x \geq 1$ and $a > 0$

$$g(x,a) = (x+1)\log\left(1+\frac{1}{x}\right) - 1 - \frac{1}{2(x+a)}. \tag{2.18}$$

By differentiation with respect to x successively, we find that for $x \geq 1$ and $a > 0$

$$g'(x,a) = \log\left(1+\frac{1}{x}\right) - \frac{1}{x} + \frac{1}{2(x+a)^2} \tag{2.19}$$

and

$$g''(x,a) = \frac{(3a-1)x^2 + 3xa^2 + a^3}{(x^3+x^2)(x+a)^3}. \tag{2.20}$$

Now from (2.20) We get

$$g''(x,\alpha) = \frac{\frac{1}{3}x + \frac{1}{27}}{(x^3+x^2)(x+\frac{1}{3})^3} > 0,$$

where α is as given in (2.17). Hence, $x \to g'(x,\alpha)$ is strictly increasing on $[1,\infty)$. But since $\lim_{x\to\infty} g'(x,\alpha) = 0$, we have $g'(x,\alpha) < 0$ for $x \geq 1$. This implies that $x \to g(x,\alpha)$ is strictly decreasing on $[1,\infty)$. From the fact that $\lim_{x\to\infty} g(x,\alpha) = 0$, we obtain $g(x,\alpha) > 0$ for $x \geq 1$, proving the left inequality of (2.16). From (2.20) we find that

$$g''(x,\beta) = \frac{a_0x^2 + a_1x + a_2}{(x^3+x^2)(x+\beta)^3}, \tag{2.21}$$

where $a_0 = -0.1169509...$, $a_1 = 0.25992524...$ and $a_2 = 0.02502972....$ It is easy to see from (2.21) that for $x \geq 3$ $g''(x,\beta) < 0$, that is , $x \to g'(x,\beta)$ is strictly decreasing on $[3,\infty)$. Since $\lim_{x\to\infty} g'(x,\beta) = 0$, this yields $g'(x,\beta) > 0$ for $x \geq 3$. But this means $x \to g(x,\beta)$ is strictly increasing on $[3,\infty)$. Since $\lim_{x\to\infty} g(x,\beta) = 0$, we get $g(x,\beta) < 0$ for $x \geq 3$. A simple calculation gives $g(1,\beta) < 0$ and $g(2,\beta) < 0$. Therefore, for any positive integer n we have $g(n,\beta) < 0$. This proves the right inequality of (2.16). From the right of (2.16) we get

$$\beta \leq \lim_{n\to\infty} \frac{1}{2\left[(n+1)\log\left(1+\frac{1}{n}\right) - 1\right]} - n.$$

It is easy to evaluate this limit and to show that has value 1/3, hence we have $\beta \leq 1/3$. Similarly, from the left inequality of of (2.16) we get for all positive integers n

$$\alpha \geq \lim_{n\to 1} \frac{1}{2\left[(n+1)\log\left(1+\frac{1}{n}\right) - 1\right]} - n,$$

which yields

$$\alpha \geq \frac{1}{4\log 2 - 2} - 1 = 0.2943497....$$

This proves that the constants α and β given in (2.17) are best possible. □

Theorem 2.7. *Let $n \geq 1$ be an integer. Then the following double inequality holds*

$$\exp\left(1 - \frac{1}{2n} + \frac{n}{3(n+\alpha)^3}\right) < \left(1 + \frac{1}{n}\right)^n < \exp\left(1 - \frac{1}{2n} + \frac{n}{3(n+\beta)^3}\right), \tag{2.22}$$

where

$$\alpha = 1/4 = 0.25 \quad \text{and} \quad \beta = \frac{1}{\sqrt[3]{3\log 2 - 3/2}} - 1 = 0.19949... \tag{2.23}$$

are best possible constants.

Proof. We define for $x \geq 1$ and $t > 0$

$$\phi(x,t) = \log(1 + 1/x) - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{3(x+t)^3}. \tag{2.24}$$

Differentiation of (2.24) with respect to x gives

$$\phi'(x,t) = \frac{(1-4t)x^3 - 6t^2x^2 - 4t^3x - t^4}{x^3(x+1)(x+t)^4}. \tag{2.25}$$

Hence, we find that

$$\phi'(x,1/4) = -\frac{24x^2 + 4x + 1}{64(x+1)^2(x+1/4)^4} < 0,$$

so that $x \to \phi(x,1/4)$ is strictly decreasing on $(1,\infty)$. Since $\lim_{x\to\infty} \phi(x,1/4) = 0$, this leads to $\phi(x,1/4) > 0$, and the proof of the left of (2.22) follows from a simple calculation. Similarly, from (2.25) we obtain that

$$\phi'(x,\beta) = \frac{c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3}{x^3(x+1)(x+\beta)^4}, \tag{2.26}$$

where $c_0 = 0.59609...$, $c_1 = 1.46934...$, $c_2 = 0.97345...$, and $c_3 = 0.20203...$, and β is as given in (2.23). Thus, we conclude that $x \to \phi(x,\beta)$ is strictly increasing for $x \geq 2$. But since $\lim_{x\to\infty} \phi(x,\beta) = 0$, we get $\phi(x,\beta) < 0$ for $x \geq 2$. An easy computation gives $\phi(1,\beta) = 0$, so that we get for all positive integers n, $\phi(n,\beta) \leq 0$. This finishes the proof of the right-hand inequality in (2.22) by the help of (2.24). Now from the right-hand inequality in (2.22)

$$\beta \leq \lim_{n\to 1} \left(\frac{1/3}{\log(1+1/n) - 1/n + 1/(2n)}\right)^{1/3} - n,$$

giving

$$\beta \leq \frac{1}{\sqrt[3]{3\log 2 - 3/2}} - 1.$$

By the same way we get from the left hand inequality of (2.22) that

$$\alpha \geq \lim_{n\to\infty} \left(\frac{1/3}{\log(1+1/n) - 1/n + 1/(2n)} \right)^{1/3} - n.$$

It is not difficult to prove that this limit goes to $1/4$ as x goes to ∞. These prove that the constants α and β given in (2.23) are best possible. □

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