On Hadwiger's results concerning Minkowski sums and isoperimetric inequalities for moments of inertia

G. KEADY
University of Western Australia

Abstract. Consider convex plane domains $D(t) = (1-t)D_0 + tD_1$, $0 \leq t \leq 1$. We first prove that the 1/4-power of the polar moment of inertia about the centroid of $D(t)$ is concave in t. From this we deduce some isoperimetric inequalities.

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1 Introduction

Although we will refer to results proved for domains in $\mathbb{R}^n$, our focus in this paper will be on domains in the plane, $n = 2$. Numerous domain functionals have been studied: torsional rigidity and various capacities are amongst those arising in physical problems: see [1, 18] for example. Our main focus in this paper will be various moments of inertia, and a large part of the paper will be re-deriving results originally established in [9].

Let D be a domain (an open connected set) in the plane. The focus in this work will be purely geometric functionals of D, for example,

- perimeter L, area A,
- centroids, Steiner curvature centroids,
- as well as various moments of inertia.

We will consider moments of inertia both for mass distributed uniformly over the region, and also – to a much less extent, and only for its uses elsewhere in these notes – for mass distributed uniformly over the boundary of the region. We will consider polar moments of inertia taken about both the Steiner centroid, denoted I_s, and the ordinary centroid, denoted I_c. I_c is treated in [9]. Several proofs are simpler when the body is 'with centre'. All these quantities are homogeneous geometric domain functionals. Definitions of terms will be given beginning in §3.

One theme of the notes is the connections between some Brunn-Minkowski-style results and isoperimetric inequalities. In particular, the result concerning the 1/4-concavity of the area polar moment of inertia about the Steiner centroids, I_s for Minkowski sums, appears to be equivalent to three isoperimetric inequalities.

- One is due to Polya, in the case $n = 2$,
$$\frac{2I_c}{\pi} \leq (\frac{L}{2\pi})^4 \tag{1.1}$$
[9], in giving a different proof, also generalized the result to n dimensions. The statement involving I_c, rather than I_s is deliberate. There are several ways to prove this. Our proof in §10 will depend on obtaining the result for bodies with centre, using $I_s = I_O = I_c$.

- Another, proved for domains with centre, is
$$(\frac{2I_0}{\pi})^3 \leq (\frac{I(\partial D)}{2\pi})^4. \tag{1.2}$$
Here $I(\partial D)$ denotes the polar moment of inertia of a uniform mass distribution along the boundary ∂D, the moment being taken about the centre O. Closely related results were proved for such domains in [15].

- The final one involves a functional, denoted Z below, appears not to have been studied before.

The structure of the paper is as follows.

- In §2 we present Polya's isoperimetric inequality, (1.1). Its proof, with a starting point similar to that in [9], is a goal of the paper.

- §3,4,5 are needed for the proofs in §7. §6 presents examples showing what is provable, and, perhaps, excusing some of the technicalities in the proofs in §7

- In §7 we present a proof of the result of [9] that the polar moment about the centroid $I_c(D(t))$ is, under Minkowski addition, 1/4-concave.
Theorem. (Hadwiger [9]) *For convex domains in n dimensions, $I_c(D(t))^{1/(n+2)}$ is concave in t for $0 \leq t \leq 1$.*
The starting point in our proof is the Prekopa-Leindler inequality, but, other than that, most elements of the proof parallel that of [9] 50 years ago.

- Support functions are introduced in §8. These are essential tools in quite separate applications in §9 and in §10.

- In §9 we use *central symmetrisation*, also called *Blaschke symmetrisation* to reduce the problem of establishing (1.1) to that of establishing it for convex sets with centre. This form of symmetrisation is appropriate for treating inequalities containing L. (As an aside, a proof of (1.1) is then presented, requiring the use of a generalized central symmetrisation. For us, this is an aside, as our goal is a unified derivation of both (1.1) and (1.2).)

- At this point, we make, in §10, a separate start. There are various useful representations of polar moments of inertia in terms of the support function of a convex domain. In §11, we use these representations with domains D_0, D_1, and hence $D(t)$, with their Steiner centroids at the origin, so $I_s = I_O$. For sets for which $I_s(D(t))$ are 1/4-concave, we establish

– an inequality like (1.1) with I_s replacing the I_c in that displayed, and

– inequality (1.2).

(There are open questions. For example, we do not know whether, for all convex sets, inequality (1.1) would be always true if I_c were to be replaced with I_s. However, we do not need the answer to this here.)

– Finally, at the end of §11, we turn to a simultaneous proof of inequalities (1.1) and (1.2). For inequality (1.1) we now merely require that it be established for convex sets with centre. For these $I_c = I_s$. We have that I_c is 1/4-concave by Hadwiger's argument. Both our inequalities are established, for centrally-symmetric convex domains, on combining this with the preceding support function result.

We haven't seen inequality (1.2) elsewhere, though some that resemble it are available. See, for example, [15], and other references mentioned near the end of §11. There are proofs of inequality (1.1) not requiring any references to Steiner centroids. Nevertheless we like the unified derivation of both inequalities simultaneously.

Some of the results here, and many conjectures, were suggested by computational experiments and computational results for simple shapes. All the functionals can be calculated exactly for any convex polygon. Furthermore, the convex hull routines of Maple and of Mathematica make the numerical calculation of Minkowski sums of polygons very easy. Supplements to this paper, including the Maple and Mathematica codes, are available via
`http//www.maths.uwa.edu.au/~keady/papers.html`

2 $2I_c/\pi \leq (L/(2\pi))^4$

For domains in the plane, Polya proved. inequality (1.1)

$$\frac{2I_c}{\pi} \leq (\frac{L}{2\pi})^4$$

Equality is attained only for disks. Hadwiger [9] gave a different proof, one which generalizes to $\mathbb{R}^n$. We remark that a combination of this (1.1) and another easier-to-prove inequality, represents a refinement of the classical (A, L), area-perimeter, isoperimetric inequality:

$$(\frac{A}{\pi})^2 \leq \frac{2I_c}{\pi} \leq (\frac{L}{2\pi})^4. \tag{2.1}$$

We first note that connectedness is essential for inequality (1.1). For, if we let D be the union of two equal disjoint disks, symmetically placed either side of the origin so that the centroid is at the origin, we have a counterexample. By taking the components further apart we can increase I_c indefinitely, while L stays fixed. However, on joining the disks by a straight line the perimeter increases and this dumbell shaped domain does satisfy inequality (1.1).

Polya's proof begins with conformal mapping, and requires D to be simply connected. However, the argument establishes inequality (1.1) for simply-connected domains. An argument now shows that it must be true for multiply-connected ones too.

Theorem 1 *Let D_m be a multiply connected domain, and D_s be the smallest simply connected domain with $D_m \subseteq D_s$. Then, if inequality (1.1) holds for D_s, it also holds for D_m.*

Proof. Denote the centroids of D_m and D_s by z_m and z_s respectively, and their perimeters by L_m and L_s. We have $L_s \leq L_m$ (as we have removed the holes of D_m and hence that part of the perimeter associated with them). Denote polar moments of inertia about z by $I(D, z)$, and, redundantly, provide a subscript c when the z is the centroid of D. Now, from properties of z_m,

$$I_c(D_m, z_m) \leq I(D_m, z_s).$$

Starting from the fact that $D_m \subseteq D_s$

$$I(D_m, z_s) \leq I_c(D_s, z_s) \leq \frac{\pi}{2}(\frac{L_s}{2\pi})^4 \leq \frac{\pi}{2}(\frac{L_m}{2\pi})^4.$$

Combining these two inequalities establishes that inequality (1.1) is true for multiply-connected domains.

In our approach to proving inequality (1.1) we will first establish it for convex domains.

Theorem 2 *Let D be a domain, and D_c be the convex hull of D. Then, if inequality (1.1) holds for D_c, it also holds for D.*

Proof. Denote the centroids of D and D_c by z_g and z_c respectively, and their perimeters by L_g and L_c. We have $L_c \leq L_g$. Now, from properties of z_g,

$$I_c(D, z_g) \leq I(D, z_c).$$

Starting from the fact that $D \subseteq D_c$

$$I(D, z_c) \leq I_c(D_c, z_c) \leq \frac{\pi}{2}(\frac{L_c}{2\pi})^4 \leq \frac{\pi}{2}(\frac{L_g}{2\pi})^4.$$

Combining these two inequalities establishes that inequality (1.1) is true for general domains.

3 Minkowski sums

3.1 Definitions

Let D_0 and D_1 be subsets of $\mathbb{R}^n$. The *Minkowski sum of D_0 and D_1* is

$$D_0 + D_1 := \{x_0 + x_1 | x_0 \in D_0,\ x_1 \in D_1\}\ .$$

We also define *dilations* of D,

$$tD := \{tx | x \in D\}.$$

The *convex combination of D_0 and D_1* is defined

$$D(t) := (1-t)D_0 + tD_1\ .$$

We will consider various families $\mathcal{U}$ of subsets of $\mathbb{R}^n$ which are closed under Minkowski sums and dilations. All these families of subsets are a commutative additive semigroups under Minkowski summation, with the identity $\{0\}$. These families include the following

$$\mathcal{K} := \{D | D \text{ is convex}\}.$$

With u a unit vector, we define the symmetrised sets

$$\mathcal{S}y(u) := \{D | \text{ if } x \in D \text{ then } x + \tau\langle u, x\rangle u \in D, \ -2 \le \tau \le 0\}$$

With $\{u_i, u_j\}$ orthonormal, we define

$$\mathcal{S}y(u_i, u_j) := \mathcal{S}y(u_i) \cap \mathcal{S}y(u_j)$$

The centrally symmetric sets are defined by

$$\mathcal{C} := \{D | \text{ if } x \in D \text{ then } \ -x \in D\}$$

The star-shaped sets are defined by

$$\mathcal{S}t := \{D | \text{ if } x \in D \text{ then } \ \tau x \in D, \ 0 \le \tau \le 1\}$$

Given a family $\mathcal{U}$, we define

$$\mathcal{U}_+ := \{D | D \in \mathcal{U} \text{ and } D \subset \{x_n \ge 0\}\}$$

More generally, we define

$$\mathcal{U}_+(u) := \{D | D \in \mathcal{U} \text{ and } D \subset \{x | \langle x, u\rangle \ge 0\}\}$$

so that, in particular $\mathcal{U}_+ = \mathcal{U}_+(e_n)$.

3.2 The context, and foundations

Borell [4] proved, in two dimensions, and denoting the torsional rigidity by S,

$$S(D(t))^{1/4} \text{ is concave in } t \text{ for } 0 \le t \le 1 \ \text{ for } D_0, D_1 \in \mathcal{K}$$

Another result of Borell's is that the transfinite diameter $\overline{r}(D(t))$ is concave. There is an obvious similarity with the familiar Brunn-Minkowski Theorem.

The results on torsional rigidity and on electrostatic capacity necessarily involve proofs from analysis. One function of this paper is to use current analytical inequalities - specifically the Prekopa-Leindler inequality - to rederive Hadwiger's results on moments of inertia. The rest of this note is only concerned with geometric domain functionals, not with the elastic torsion problem or electrostatic capacity. Nevertheless, we note that the moment of inertia functional is sandwiched between (appropriate numeric multiples of) the two 1/4-concave functionals, S and $\overline{r}^4$: see [18], page 10. Actually the area squared, which is - by our next theorem - also 1/4 concave is a better lower bound on I_c than is S.

Theorem 3 General Brunn-Minkowski *Let $0 < t < 1$ and D_0 and D_1 be nonempty bounded measurable sets in $\mathbb{R}^n$ such that $D(t)$ is also measurable. Then*

$$\text{Volume}(D(t))^{1/n} \ge (1-t)\text{Volume}(D_0)^{1/n} + t\text{Volume}(D_1)^{1/n} \tag{3.1}$$

(That is, $D(t)$ is concave in t for $0 \le t \le 1$.)
Specializing to $n = 2$, this becomes

$$\text{Area}(D(t))^{1/2} \textit{ is concave in } t \text{ for } 0 \le t \le 1 \ . \tag{3.2}$$

See [7], Theorem 4.1. The paper [7] also explains the equivalence of this and the Prekopa-Leindler Inequality.

Theorem 4 *Let $0 < \lambda < 1$ and let f_0, f_1, and h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying*

$$h\left((1-t)x+ty\right) \geq f_0(x)^{1-t} f_1(y)^t, \tag{3.3}$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\, dx \geq \left(\int_{\mathbb{R}^n} f_0(x)\, dx\right)^{1-t} \left(\int_{\mathbb{R}^n} f_1(x)\, dx\right)^{t}. \tag{3.4}$$

4 Notes on homogeneous concave functions

4.1 Definitions, and introduction

A functional $F(D)$ is *homogeneous of degree k* if $F(cD) = c^k F(D)$ for all $D \in \mathcal{K}$ and $c \geq 0$. The functionals Area and torsional rigidity, S, are homogeneous of degree 2 and 4 respectively. If $F(D)$ is a functional which is homogeneous of degree k, I will call it *Minkowski-polynomial* if $F(D(t))$ is polynomial of degree k in the variable t. A functional $F(D)$ is said to be *nonnegative* if $F(D) \geq 0$ for all D, and *monotone* if $D_1 \subseteq D_2$ implies that $F(D_1) \leq F(D_2)$. A functional $F(D)$ is said to be *translation-invariant* if $F(c+D) = F(D)$ for all D and all $c \in \mathbb{R}^2$.

The area and torsional-rigidity are but two of many nonnegative, monotone, translation-invariant homogeneous domain functionals. Denote by $I_c(D)$ the polar moment of inertia (about the centroid z_c). This is homogeneous of degree 4. (Furthermore, although I_c itself is not Minkowski-polynomial, the product AI_c of the area with I_c is Minkowski-polynomial of degree 6.)

4.2 An elementary result

$I_c(D(t)$ and the other functionals treated in this note are homogeneous. This may help in establishing the concavity properties. This little lemma gets used (though not dignified by being called a Lemma) in [7].

Lemma 1 (Homogeneity Lemma.) *Consider domains in a family closed under Minkowski sums and under dilations. If F is positive and homogeneous of degree 1*

$$F(sD) = sF(D)\ \forall s > 0, D\ ,$$

and quasiconcave

$$F(D(t)) \geq \min(F(D(0)), F(D(1)))\ \forall 0 \leq t \leq 1\ \forall D_0, D_1, \tag{4.1}$$

then it is concave:

$$F(D(t)) \geq (1-t)F(D(0)) + tF(D(1))\ \forall 0 \leq t \leq 1\ .$$

Proof. See [7]. Replace D_0 by $D_0/F(D_0)$, D_1 by $D_1/F(D_1)$. Using the homogeneity of degree 1, and applying (4.1) we have

$$F((1-t)\frac{D_0}{F(D_0)}+t\frac{D_1}{F(D_1)})\geq 1\ .$$

With

$$t=\frac{F(D_1)}{F(D_0)+F(D_1)}\ ,\qquad \text{so } (1-t)=\frac{F(D_0)}{F(D_0)+F(D_1)}\ ,$$

the last inequality on F becomes

$$F\left(\frac{D_0+D_1}{F(D_0)+F(D_1)}\right)\geq 1\ .$$

Finally, using the homogeneity we have

$$F(D_0+D_1)\geq F(D_0)+F(D_1)\ ,$$

and using homogeneity again,

$$F((1-t)D_0+tD_1)\geq (1-t)F(D_0)+tF(D_1)\ ,$$

as required.

5 Power concave functions

When $\alpha>0$, we say that a nonnegative function f is *α-concave* if f^α is concave. See also [7]§9. We say that f is *0-concave*, or *log-concave*, if $\log(f)$ is concave. (For $-\infty<\alpha<0$, the notation may not yet be standardised. However, in some definitions, when $\alpha<0$, it is said that a nonnegative function f is *α-concave* if f^α is convex.) A nonnegative function f is said to be *$-\infty$-concave*, or *quasiconcave*, if $f((1-s)t_0+st_1)\geq\min(f(t_0),f(t_1))\forall s\in[0,1]$.

Here are some properties.

1. If a nonnegative f is α-concave, then f is β-concave for all $\beta\leq\alpha$.
2. If a nonnegative f is twice continuously differentiable, f is α-concave iff $ff''+(\alpha-1)f'^2\leq 0$.
3. For any $\alpha\geq 1$, the set of nonegative α-concave functions forms a (convex) cone.
4. If $\alpha\geq 0$, $\beta\geq 0$, and f is α-concave, g is β-concave, then the product fg is γ-concave where $\gamma^{-1}=\alpha^{-1}+\beta^{-1}$.
   The product of log-concave functions is log-concave.

6 Lead-in examples

Consider two equal equilateral triangles, D_0 and D_1, each with a median on the y-axis. D_0 points downward, and has vertices

$$A_0:(1,y_{c,0}+\frac{1}{\sqrt{3}}),\qquad B_0:(-1,y_{c,0}+\frac{1}{\sqrt{3}}),\qquad (0,y_{c,0}-\frac{2}{\sqrt{3}}).$$

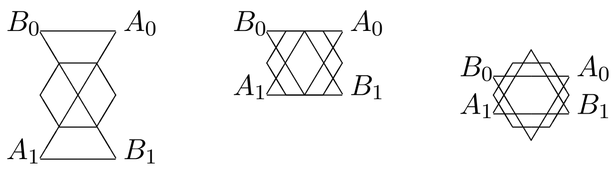

Figure 1: The Minkowski sum of equilateral triangles D_0 and D_1 in various positions. $(D_0 + D_1)/2$ is a regular hexagon, in fact in the same position in the leftmost and rightmost diagrams.

D_1 points upward, and has vertices

$$A_1 : (-1, y_{c,1} - \frac{1}{\sqrt{3}}), \qquad B_1 : (1, y_{c,1} - \frac{1}{\sqrt{3}}), \qquad (0, y_{c,1} + \frac{2}{\sqrt{3}}).$$

See Figure 1.

The set $H = (D_0 + D_1)/2$ is a regular hexagon, with a pair of sides parallel to the x-axis, and its centre at $(0, (y_{c,0} + y_{c,1})/2)$.

We have

$$\text{area}(D_0) = \sqrt{3} = \text{area}(D_1) \text{ while } \text{area}(H) = \frac{3\sqrt{3}}{2}.$$

The fact that

$$\text{area}(H) \geq \min(\text{area}(D_0), \text{area}(D_1))$$

accords with the quasiconcavity of area$(D(t))$ as required by the classical Brunn-Minkowski Theorem.

We next consider moments about the $y = 0$ axis, specifically

$$I_{22}(D) = \int_D y^2$$

Of course this functional is not translation-invariant. Let's look at a few instances.

- $y_{c,0} = 2/\sqrt{3}$ and $y_{c,1} = -2/\sqrt{3}$: In this situation, the vertices of the triangles on the $x = 0$ axis are both at the origin.

  $$I_{22}(D_0) = \frac{3\sqrt{3}}{2} = I_{22}(D_1) \text{ while } I_{22}(H) = \frac{5\sqrt{3}}{16}$$

  Here, $I_{22}(D(t))$ is definitely not quasiconcave.

- $y_{c,0} = 2/\sqrt{3}$ and $y_{c,1} = 1/\sqrt{3}$: In this situation, both of the triangles lie in the upper half-space (and the origin is on the boundary of each of them).

  $$I_{22}(D_0) = \frac{3\sqrt{3}}{2},\ I_{22}(D_1) = \frac{\sqrt{3}}{2}, \text{ while } I_{22}(H) = \frac{41\sqrt{3}}{16}$$

  With a bit more calculation, this time it can be shown that $I_{22}(D(t))$ is quasiconcave. This is consistent with Hadwiger's Theorem 2.

- Our main results concern moments about the centroid, and, in particular, we will need to consider moments when the centroid of $D(t)$ lies on $y = 0$. So, for our final example here, consider:
  $y_c(H) = 0$ with $y_{c,0} = 0$ and $y_{c,1} = 0$: In this situation,

$$I_{22}(D_0) = \frac{\sqrt{3}}{6},\ I_{22}(D_1) = \frac{\sqrt{3}}{6},\ \text{while } I_{22}(H) = \frac{5\sqrt{3}}{16}$$

  There are possibly two separate uses for this example.

  - $I_{22,c}$ is quasi-concave.
  - There is perhaps too much symmetry in this $t = 1/2$ case to truly represent enough of Hadwiger's Theorem 3. For general t, the centroids of D_0 and D_1 are not at the origin (while that of $D(t)$ is). Nevertheless, our example is consistent with the result that this $I_{22}(D(t))$ is again quasiconcave.

7 Hadwiger's proofs: $1/4$-concavity of I_c

Hadwiger's approach begins considering sets in $\mathcal{U}_+$.

Theorem 5 *Define, for each unit vector u, and sets D in $\mathcal{U}_+(u)$*

$$T_+(D, u) = \int_D \langle u, x\rangle^2$$

then $T_+(D(t), u)$ is 1/(n+2)-concave.

In view of the counter-examples to concavity when one takes moments of inertia about $y = 0$ when the sets are not restricted to being in the upper half-plane, the result is striking.

We will begin with our own proof based on the Prekopa-Leindler Theorem 4, and after this give Hadwiger's proof.
Proof of Theorem 5, when D_0 and D_1 are convex, from the Prekopa-Leindler inequality. In Theorem 4, Theorem 7.1 of [7], we are required to define functions on the whole of $\mathbb{R}^2$ and, to do this usefully, we replace the y^2 with y_+^2, i.e. a function which is 0 for $y \leq 0$. It is essential that the sets D_0 and D_1 are in the upper half-plane. The function $z = (x, y) \mapsto y^2$ is convex, which doesn't help, but, *in* $y > 0$, it is also log-concave.
(1) The function $z = (x, y) \mapsto y_+^2 =: f(z)$ is log-concave on the whole space. For consider $f(z)$ and $z_0 = (x_0, y_0)$, $z_1 = (x_1, y_1)$. Without loss of generality, assume $y_0 \leq y_1$. To establish log-concavity, we need to show

$$f(\frac{z_0 + z_1}{2}) \geq \sqrt{f(z_0)f(z_1)}.$$

Now, if either $y_0 \leq 0$ or $y_1 \leq 0$ this is trivially satisfied, so assume $0 < y_0 \leq y_1$. Then by the AGM inequality

$$f(\frac{z_0 + z_1}{2}) = (\frac{y_0 + y_1}{2})^2 \geq y_0 y_1 = \sqrt{f(z_0)f(z_1)},$$

as required. More generally, let $z_t = (1 - t)z_0 + t z_1$. Then

$$f(z_t) \geq f(z_0)^{1-t} f(z_1)^t.$$

(2) The characteristic functions χ of the convex sets satisfy

$$\chi_{D(t)}((1-t)(z_0)+t(z_1)) = \chi(D_0)(z_0)^{1-t}\chi(D_1)(z_1)^t$$

(i.e. equality, not merely that the lhs is greater than or equal to the rhs).
Items (1) and (2) combine to give

$$f(z_t)\chi_{D(t)}(z_t) \geq (f(z_0)\chi_{D_0}(z_0))^{1-t}(f(z_1)\chi_{D_1}(z_1))^t.$$

Now an application of the Prekopa-Leindler Theorem yields the log-concavity of $I_{22}(D_+(t))$. The homogeneity then gives its (1/4)-concavity.

Though we do not yet see any application, we note that the same argument applies to any function $f(z) = (y_+)^\alpha$ for any $\alpha > 0$. In n dimensions we would get $1/(n+\alpha)$-concavity.

Hadwiger's proof of Theorem 5. The starting point for this is some manipulation to reduce this problem to an application of the Volume Brunn-Minkowski Theorem in $(n+2)$ dimensions. In our account now, we will specialise to $n = 2$ dimensions.

Lemma 2 *Let u be a unit vector. Define map from the two-dimensional $D \in \mathcal{U}_+(u)$ to a $\mathbb{R}^4$ by*

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + (u_1x_1+u_2x_2)(\sigma_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \sigma_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}),\ 0 \leq \sigma_3 \leq 1,\ 0 \leq \sigma_4 \leq 1$$

Associated with the plane domains $D(t)$ define 4-dimensional domains $\tilde{D(t)}$.
(i)

$$T_+(D(t), u) = \text{Volume}(\tilde{D(t)})$$

(ii) The tilde-construction and Minkowski summation commute

$$\tilde{D(t)} = (1-t)\tilde{D}_0 + t\tilde{D}_1$$

Proof. For ease of exposition, we choose to set $u = e_2$.
(i)

$$\tilde{x} = xe_1 + ye_2 + y(\sigma_3e_3 + \sigma_4e_4),\ 0 \leq \sigma_3 \leq 1,\ 0 \leq \sigma_4 \leq 1$$

Now the volume of $\tilde{D}$ is

$$\begin{aligned} \text{Volume}(\tilde{D}) &= \int\int_D \int_0^1 y\, d\sigma_3 \int_0^1 y\, d\sigma_4 dy dx \\ &= T_+(D) = I_{22}(D) \end{aligned}$$

(ii) The key observation that establishes (ii) is that at fixed σ_3 and σ_4, the map from x to $\tilde{x}$ is linear.

Hadwiger then establishes Theorem 5 using Lemma 2.

We expect that if we were to replace I_{22} by I_O (i.e. the log-concave function y^2 by the function r^2 which appears to have no useful concavity properties) that there would be no result.

Although Theorem 5 makes a start, we have some way to go before we get to establish the (1/4)-concavity of $I_c(D(t))$. The two main steps are as follows.

(i) We will extend the result from $\mathcal{U}_+$ to all of $\mathcal{U}$. See Lemmas 4 and 5.

(ii) Having established the result for some class of domains $\mathcal{U}$ for all directions u, we will obtain the polar-moment of inertia result on $I_c(D(t))$. See Theorem 3

There is, though, one case where (i) is straightforward, and we treat that now. Suppose that D_0 and D_1 are in $\mathcal{C}$, i.e. are centrally symmetric, with both being centred on the origin. Now $\mathcal{C}$ is closed under Minkowski sums and under dilation, with $D(t) \in \mathcal{C}$ when both D_0 and D_1 are in $\mathcal{C}$. Now write $D = D_+ \cup D_-$, with $D_+ = D \cap \{y \geq 0\}$ and $D_- = D \cap \{y \leq 0\}$. Now

$$I_{22}(D) = I_{22}(D_+) + I_{22}(D_-)$$

and,

$$I_{22}(D_+) = I_{22}(D_-) \text{ using } D \in \mathcal{C}$$

Now, by Lemma 2, $I_{22}(D_+)$ is (1/4)-concave, and hence so to is $I_{22}(D)$. We also note, in preparation for (ii), that we could use any direction u, not merely the direction e_2 used as in the example, and indicated in the subscripting I_{22}.

We will return to the general case later, but we now turn to task (ii), getting the result for $I_c(D)$ not merely the $I(D, u) = T(D, u)$.

Lemma 3 *Suppose that $T(D, u) = u^T \mathcal{M}(D)u$, with $\mathcal{M}(D)$ a n by n positive definite matrix function of D only (i.e. not u). It clearly satisfies*

$$T(D, -u) = T(D, u).$$

Suppose also that each entry of $\mathcal{M}(D)$ is homogeneous to the same power so that clearly $T(D, u)$ is homogeneous to this power.
Suppose that $T(D(t))$ is, under Minkowski-sums, α-concave for some $\alpha > 0$. Then $i(D) = \mathrm{trace}(\mathcal{M}(D))$ is homogenous to the same power, and

$$i(t) = \mathrm{trace}(\mathcal{M}(D(t)))$$

is also α-concave.

Proof. It suffices to allow D_0 and D_1 to have the property that $i(0) = 1 = i(1)$. The homogeneity of $i(D)$, which is immediate from the hypotheses on the matrix $\mathcal{M}(D)$, ensures that we merely need to establish that $i(t)$ is quasiconcave after which an the Homogeneity Lemma will give that it is power-concave. We will have established this once we show $i(t) \geq 1$ (for all t in $[0, 1]$).

Now $\mathcal{M}$ is positive definite, so its trace, the sum of its eigenvalues is positive, and

$$\mathrm{trace}(\mathcal{M}(D)) = \sum_{j=1}^{n} T(D, u_j)$$

for any orthonormal basis $\{u_j\}$. Our goal is to choose an orthornormal basis which will help in establishing $i(t) \geq 1$.

Define

$$f(u) = T(D_0, u) - T(D_1, u).$$

As elsewhere, we specialise to the 2 by 2 matrix case appropriate to domains in the plane. (The general case is treated in Hadwiger's paper.) Then f is 'defined on a circle'. We will show that there are directions at right angles so that $f(u_1) = f(u_2)$, and to do this, define

$$g(\theta) = f\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - f\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Using $f(-u) = f(u)$ we find

$$g(\theta) + g(\theta + \frac{\pi}{2}) = 0.$$

Hence g is sometimes nonnegative and sometimes nonpositive, and hence there is a θ_* at which $g(\theta_*) = 0$. This ensures that we have the directions at right angles so that $f(u_1) = f(u_2)$.

We now translate this into $T(D(t), u)$ notation. We have two equations from the (scaling) hypotheses on D_0 and D_1, and, in particular, these give $f(u_1) + f(u_2) = 0$. Combining this with our choice of u_1, u_2 with $f(u_1) = f(u_2)$ forces $f(u_1) = 0 = f(u_2)$. Thus

$$\begin{aligned} T(D_0, u_1) + T(D_0, u_2) &= 1 \\ T(D_1, u_1) + T(D_1, u_2) &= 1 \\ T(D_0, u_1) &= T(D_1, u_1) \\ T(D_0, u_2) &= T(D_1, u_2) \end{aligned}$$

Now we have from the quasiconcavity of $T(D(t), u_j)$,

$$\begin{aligned} T(D(t), u_1) &\geq T(D_0, u_1)\ (\ = T(D_1, u_1)\ ) \\ T(D(t), u_2) &\geq T(D_0, u_2)\ (\ = T(D_1, u_2)\ ) \\ i(t) &\geq i(0) = 1\ (\ = i(1)\ ) \end{aligned}$$

where the last inequality is merely the result of summing the two preceding ones. This establishes the quasiconcavity of $i(t)$ and the proof is complete.

For centrally-symmetric sets, Theorem 5 and Lemma 3 in combination establish Hadwiger's Theorem that, for such sets, $I_c(D(t))^{1/4}$ is concave. We now turn to extending this so that it applies more generally, and not merely to bodies 'with centre'. The key ingredient in this will be Theorem 6, whose proof will depend on Lemmas 4 and 5. (The results of Lemmas 4, 5 and Theorem 6 would be proven if we had established them for domains symmetrised about the axis $x = 0$ as this symmetrisation preserves y_c and $I_{22}(D)$. However, this observation is not needed for our proofs.)

Lemma 4 *Fix t, $0 < t < 1$. Choose axes so that $y = 0$ passes through $D(t)$. (In the application, in [9], $y_c(D(t)) = 0$.)*
For each domain D, define

$$D_+ = D \cap \{y > 0\}, \qquad D_- = D \cap \{y < 0\}.$$

Define

$$D_0(\tau) = D_0 + \{(0, \tau/(1-t))\},\ D_1(\tau) = D_1 - \{(0, \tau/t)\}$$

so that, again, for all τ*,*

$$D(t) = (1-t)D_0(\tau) + tD_1(\tau)$$

Define

$$\begin{aligned} \xi(\tau) &= \frac{I_{22}((D_0(\tau))_+)}{I_{22}((D_0(\tau))_-)} \\ \eta(\tau) &= \frac{I_{22}((D_1(\tau))_+)}{I_{22}((D_1(\tau))_-)}. \end{aligned}$$

Then τ *can be chosen so that* $\xi(\tau) = \eta(\tau)$*.*

Proof. When τ is large and positive, then $\xi(\tau)$ is positive infinity, and $\eta(\tau)$ is 0. Similarly, when τ is large and negative, then $\xi(\tau)$ is 0, and $\eta(\tau)$ is positive infinity. Consider monotonically increasing τ from some large negative value. Both $\xi(\tau)$ and $\eta(\tau)$ are monotonic, continuous functions. ξ is a nonincreasing function, η is a nondecreasing function. Thus, by the Intermediate Value Theorem, there exists a τ_* at which $\xi(\tau_*) = \eta(\tau_*)$.

Lemma 5 *Fix* t*,* $0 < t < 1$*. Choose axes so that* $y = 0$ *passes through* $D(t)$*. (In the application, in [9],* $y_c(D(t)) = 0$*.)*
For each domain D*, define*

$$D_+ = D \cap \{y > 0\}, \qquad D_- = D \cap \{y < 0\}.$$

Suppose that

$$D(t) = (1-t)D_0 + tD_1$$

with

$$\frac{I_{22}((D_0(\tau))_+)}{I_{22}((D_0(\tau))_-)} = \frac{I_{22}((D_1(\tau))_+)}{I_{22}((D_1(\tau))_-)} \ (= \mu).$$

Then

$$I_{22}(D(t))^{1/4} \geq (1-t)I_{22}(D_0)^{1/4} + tI_{22}(D_0)^{1/4}$$

Proof. From Theorem 5

$$\begin{aligned} I_{22}((D(t))_+)^{1/4} &\geq (1-t)I_{22}((D_0)_+)^{1/4} + tI_{22}((D_1)_+)^{1/4} \\ I_{22}((D(t))_-)^{1/4} &\geq (1-t)I_{22}((D_0)_-)^{1/4} + tI_{22}((D_1)_-)^{1/4} \end{aligned}$$

Thus, using properties of Minkowski sums,

$$\begin{aligned} I_{22}(D(t)) &\geq I_{22}((D(t))_+) + I_{22}((D(t))_-) \\ &\geq E_-^4 + E_+^4 \end{aligned} \tag{7.1}$$

where

$$\begin{aligned} E_- &= ((1-t)I_{22}((D_0)_+)^{1/4} + tI_{22}((D_1)_+)^{1/4}) \\ E_+ &= ((1-t)I_{22}((D_0)_-)^{1/4} + tI_{22}((D_1)_-)^{1/4}) \\ E &= ((1-t)I_{22}((D_0))^{1/4} + tI_{22}((D_1))^{1/4}). \end{aligned}$$

Now

$$I_{22}((D_j)_-) = \frac{I_{22}(D_j)}{1+\mu},\ I_{22}((D_j)_+) = \frac{\mu I_{22}(D_j)}{1+\mu}.$$

Hence the expressions E satisfy

$$E_+ = (\frac{\mu}{(1+\mu)})^{1/4} E,\ E_- = (\frac{1}{(1+\mu)})^{1/4} E.$$

On entering this in the expression at the right at inequality (7.1) we have

$$I_{22}(D(t)) \geq (\frac{\mu}{(1+\mu)}) E^4 + (\frac{1}{(1+\mu)}) E^4 = E^4$$

which is the result we were required to prove.

Theorem 6 *Define, for each unit vector u, and sets D in $\mathcal{U}$*

$$I_c(D,u) = \int_D \langle u, x\rangle^2 - \frac{(\int_D \langle u, x\rangle)^2}{\int_D 1}$$

then $I_c(D(t),u)$ is 1/(n+2)-concave in t.

Proof. We choose the origin at the centroid of $D(t)$. We use the two immediately preceding lemmas and the fact that, for each of D_0 and D_1, the moment taken about the lines through its centroid will be less than the moment taken about any parallel line.

Proof of Hadwiger's I_c Theorem of §1. The result follows from Theorem 6 and, with $T(D,u) = I_c(D,u)$, from Lemma 3.

We remark that Hadwiger had $y_c(D(t)) = 0$ in his applications of Lemmas 4 and 5, but, while this is needed for the I_c result, it is not needed for the actual results of the Lemmas. It is an open question whether other forms of centroids, e.g. the Steiner curvature centroid might be used with worthwhile consequences. That we are using the ordinary centroid does get use in the final sentence of Theorem 6.

8 Support functions

We use the letter p for the support function as in [6] and Santalo's book [19]. An adequate description of p is as 'the perpendicular distance from the origin to the tangent'. We will use the letter ϕ exactly as in Santalo's book, and define

$$\varphi = \phi + \frac{\pi}{2}.$$

These notes were begun with a focus on the plane case and for that φ, the slope of the tangent associated with p, is acceptable. The radius of curvature is

$$\rho = p + \ddot{p}, \quad \text{where } \dot{f} = \frac{df}{d\varphi},\ ds = \rho d\varphi\ .$$

We must have

$$\int_0^{2\pi} \rho(\varphi)\cos(\varphi)\, d\varphi = 0 = \int_0^{2\pi} \rho(\varphi)\cos(\varphi)\, d\varphi.$$

Then the area and perimeter are given by

$$A = \text{Area}(D) = \frac{1}{2}\int_0^{2\pi} p\rho\, d\varphi = \frac{1}{2}\int_0^{2\pi} (p^2 - \dot{p}^2) d\varphi, \quad (8.1)$$

$$L = \int_0^{2\pi} \rho\, d\varphi = \int_0^{2\pi} p\, d\varphi\ . \quad (8.2)$$

D is convex if and only if $\rho \geq 0$. In the case of a polygon, for example, we might interpret ρ as a nonnegative measure. We will merely be assume that the boundaries of D, and the functions p, are sufficiently smooth for any operations we perform. The set $\mathcal{S}$ of nonnegative support functions forms a cone: $\mathcal{S}$ is convex, and, if $t > 0$ and $p \in \mathcal{S}$, then $tp \in \mathcal{S}$.

We now suppose that we have two convex domains D_0 and D_1. We denote $\text{Area}(D_0) = A_0$ and $\text{Area}(D_1) = A_1$. We have the following pretty, and very well-known, result:

Lemma 6 *For convex sets D_0, D_1, the support function for $D(t)$ is given by*

$$p_t = (1-t)p_0 + tp_1 \tag{8.3}$$

In particular, the preceding lemma yields that

$$\begin{aligned} L(t) &:= L(D(t)) = (1-t)L_0 + tL_1 \ , && (8.4) \\ A(t) &:= \text{Area}(D(t)) = (1-t)^2 A_0 + 2t(1-t)A_{0,1} + t^2 A_1 \ , && (8.5) \end{aligned}$$

where the *mixed-area* $A_{0,1}$ satisfies

$$A_{0,1} := A(D_0, D_1) = \frac{1}{2} \int_0^{2\pi} (p_0 p_1 - \dot{p}_0 \dot{p}_1) \, d\varphi \ . \tag{8.6}$$

(We remark that when $D_1 = B(0, \rho_1)$ is a disk, these immediately give us the formulae for the mixed area and the equivalence of the area Brunn-Minkowski and the classical isoperimetric inequality. We will see this in §11.2.)

There are many nice properties of support functions. Here is one. See [20], p37 or the first page of [14].

Theorem 7 *If $0 \in D \subseteq \hat{D}$ then $0 \leq p \leq \hat{p}$.*

We do not use, but state:

Theorem. *Let C denote the convex hull of the union of the convex domains D_0 and D_1. Then, the support functions satisfy*

$$p_C = \max(p_{D_0}, p_{D_1}).$$

. See [5], p56.

Henceforth, we assume that the origin is in any domain D. (In §9, the origin will often be at the point of central symmetry. In §11, the origin will often be at the Steiner centroid.)

(Further general references on convex domains and their support functions include [3], [6], [8], [11], [14], [16], [20].)

9 Central, or Blaschke, symmetrisation

A major function of this section is to show that to establish inequality (1.1), it will suffice to establish it for centrally-symmetric domains. In central-symmetrisation, we start with the support function p of a convex domain, and create another convex domain D^* whose support function is given by

$$p^* = \frac{p(\varphi) + p(\varphi + \pi)}{2}.$$

Central symmetrisation preserves L. It increases I_c:

Lemma 7

$$I_c(D^*)^{1/4} \geq \frac{1}{2}(I_c(D)^{1/4} + I_c(-D)^{1/4}) = I_c(D)^{1/4}$$

Proof. Note that the support function of $-D$ is $p(\varphi + \pi)$. The result of the Lemma then follows from the 1/4-concavity of I_c.

Thus, if there were to be a domain which did not satisfy inequality (1.1), there would be one which was centrally symmetric. We will, however, in §10, show that there is no such centrally symmetric domain. A key point is that for such centrally symmetric domains, the centroid and the Steiner curvature centroid, defined soon, coincide at the origin. In notation which will be defined, for centrally-symmetric domains, $I_c = I_s$.

9.1 A side track: alternative proofs of (1.1)

There are many other symmetrisations possible. For example, at fixed φ_0 and m,

$$p^*(\varphi) := \frac{1}{2m}\sum_{j=0}^{2m-1} p(\varphi - \varphi_0 + \frac{j\pi}{m}).$$

Again, this preserves L and (again using the 1/4-concavity) increases I_c. The choice of origin φ_0 for φ is arbitrary.

The 1/4-concavity can be expressed in many ways. When, as with each D_j being a rotated version of the same initial set D_0, the $I_c(D_j)$ are all equal, we have

$$I_c(\frac{1}{m}\sum_{j=1}^{m} D_j)^{1/4} \geq \frac{1}{m}\sum_{j=1}^{m} I_c(D_j)^{1/4} = I_c(D_0)^{1/4}.$$

Taking the limit as m tends to infinity, we get that I_c is maximised at fixed L when D is a disk.

10 Expressions for I_O, etc. in terms of p

10.1 More geometry

In our usage, φ is the angle between the tangent to the curve, the boundary of D, and the x-axis. (This is as in Behnke et al. Many authors use notation which differs from our φ by $\pi/2$. These include: Santalo's ϕ as in his Fig 1.1; the θ in [6] as in his Fig 2; and the θ in [8].) We need to investigate $x(\varphi)$ and $y(\varphi)$ on the boundary of D. We have

$$\frac{dx}{d\varphi} = \rho\cos(\varphi), \qquad \frac{dy}{d\varphi} = \rho\sin(\varphi), \tag{10.1}$$

$$x\sin(\varphi) - y\cos(\varphi) = p. \tag{10.2}$$

There are numerous readily verified identities, e.g. $\rho = -\ddot{x}\sin(\varphi) + \ddot{y}\cos(\varphi)$. We have

$$\dot{p} = x\cos(\varphi) + y\sin(\varphi) + \dot{x}\sin(\varphi) - \dot{y}\cos(\varphi) = x\cos(\varphi) + y\sin(\varphi)\ , \tag{10.3}$$

and, from (10.1) and (10.3),

$$\frac{d}{d\varphi}(x^2 + y^2) = 2\rho(x\cos(\varphi) + y\sin(\varphi)) = 2\rho\dot{p} = \frac{d}{d\varphi}(p^2 + \dot{p}^2)\ ,$$

or, more simply, from squaring both (10.3) and (10.2) and adding,

$$p^2 + \dot{p}^2 = x^2 + y^2 \ . \tag{10.4}$$

We also have

$$\begin{aligned} x &= \dot{p}\cos(\varphi) + p\sin(\varphi) \\ y &= \dot{p}\sin(\varphi) - p\cos(\varphi) \end{aligned}$$

Using φ to parametrise the boundary of our sets $D(t)$,

$$\partial D(t) = \{(x_t(\varphi), y_t(\varphi))|0 \le \varphi \le 2\pi\},$$

is very convenient when dealing with Minkowski combinations. We have

$$x_t = (1-t)x_0 + tx_1, \qquad y_t = (1-t)y_0 + ty_1 \ ,$$

following from $p_t = (1-t)p_0 + tp_1$ and the corresponding relation for ρ.

10.2 Moments of inertia, etc.

We now turn to moments of inertia. Applications of Green's Theorem give

$$A = \frac{1}{2}\oint (xdy - ydx) = \frac{1}{2}\int_0^{2\pi} p\rho d\varphi \tag{10.5}$$

$$x_c A = \frac{1}{2}\oint x^2 dy \tag{10.6}$$

$$y_c A = -\frac{1}{2}\oint y^2 dx \tag{10.7}$$

$$I_o = I(0,D) = \frac{1}{4}\oint (x^2+y^2)(xdy - ydx) \tag{10.8}$$

$$= \frac{1}{4}\int_0^{2\pi} (p^2+\dot{p}^2) p\rho d\varphi \ . \tag{10.9}$$

$$I(\partial D) = \int_0^{2\pi} (x^2+y^2)\rho \, d\varphi \tag{10.10}$$

$$= \int_0^{2\pi} (p^2+\dot{p}^2)\rho d\varphi \tag{10.11}$$

Where one sees a $\rho d\varphi$, this can be written ds.

Eliminating ρ from the preceding equations using $\rho = p + \ddot{p}$ and integrating by parts gives

$$I_o = I(0,D) = \frac{1}{12}\int_0^{2\pi} (3p^4 - 6p^2\dot{p}^2 - \dot{p}^4) d\varphi \ . \tag{10.12}$$

$$I(\partial D) = \int_0^{2\pi} p(p^2 - \dot{p}^2) d\varphi \tag{10.13}$$

while the corresponding expressions for L and A were given in equations (8.2) and (8.1). Concerning equation (10.13), we remark that $I(\partial D)$ is the moment of inertia of a wire, of unit mass per unit length, in the shape of ∂D, the boundary of D. Another domain functional that will be used later is

$$Z = \frac{1}{2}\int_0^{2\pi} (3p^2 - \dot{p}^2) d\varphi \tag{10.14}$$

11 Isoperimetric inequalities derived from Brunn-Minkowski

We now turn to domains D_0 and D_1 where the Steiner curvature centroid coincides with the ordinary centroid, and we take this at the origin O. In fact, in what follows, it will suffice to take $D_1 = B(0, \rho_1)$ as the disk centered at the origin and of radius ρ_1.

11.1 The Steiner curvature centroid

The Steiner curvature centroid for a convex domain D, with support function p, is the point (x_s, y_s) where

$$x_s = -\frac{1}{\pi} \int_2^{2\pi} p(\varphi) \sin(\varphi)\, d\varphi, \qquad y_s = \frac{1}{\pi} \int_2^{2\pi} p(\varphi) \cos(\varphi)\, d\varphi.$$

We have that $(x_s, y_s) \notin \partial D$; $(x_s, y_s) \in D$. See [3], p58.

Theorem 8 *Suppose that domains D_0 and D_1 have Steiner centroids s_0, s_1 respectively. Then the Steiner centroid of $D(t)$ is $s(t) = (1-t)s_0 + t\ s_1$. In particular, if $s_0 = 0 = s_1$, then $s(t) = 0$ for all t with $0 \le t \le 1$.*

11.2 A lead-in isoperimetric inequality example, A, L

We will derive isoperimetric inequalities from Brunn-Minkowski results, and begin with illustrating this with the ordinary area Brunn-Minkowski. Take $D_1 = B(0, \rho_1)$, so that the support functions satisfy $p_t = (1-t)p_0 + t\rho_1$. Substituting this into equation (8.1) we find

$$A(t) = (1-t)^2 A_0 + (1-t)tL_0\rho_1 + t^2\pi\rho_1^2.$$

Then we find

$$\frac{A(t) - ((1-t)\sqrt{A0} + t\rho_1\sqrt{\pi})^2}{(1-t)t} = (L_0 - 2\sqrt{\pi A_0})$$

and we see that the Brunn-Minkowski 1/2-concavity of area is equivalent to the classical isoperimetric inequality $L^2 \ge 4\pi A$.

(It can also be shown that $A(D(t))/L(D(t))$ is concave. The result is in [3], but is easy to prove. One can also show, easily enough, that $\sqrt{Z(D(t))}$ is concave. Though, when $D_1 = B(0, \rho_1$, it also happens that $Z(D(t))/L(D(t))$, we do not know what is true in general. As our primary focus is on the behaviour of moments of inertia, we do not pursue matters concerning $Z(D)$ here.)

11.3 I_s, $I(\partial D)$ and L

We now prepare to perform similar calculations with moments of inertia. On substituting $p_t = (1-t)p_0 + t\rho_1$ into equation (10.12) we find:

$$\begin{aligned} I_s(t) &= (1-t)^4 I_s(D_0) + \rho_1 t(1-t)^3 I(\partial(D_0) + \\ &\quad \rho_1^2 t^2 (1-t)^2 Z(D_0) + \\ &\quad \rho_1^3 t^3 (1-t) L_0 + t^4 \frac{\pi\rho_1^4}{2} \end{aligned} \qquad (11.15)$$

Define the quadratic polynomial

$$q(t) := \frac{I(D(t)) - (I(D(0))^{1/4}(1-t) + I(D(1))^{1/4}t)^4}{t(1-t)}. \tag{11.16}$$

Theorem 9 *Suppose that D_1 is the disk $B(0,\rho_1)$, that $D_0 \in \mathcal{K}$, and that, for $0 \le t \le 1$ the Steiner curvature of $D(t)$ coincides with its ordinary centroid, each being at the origin. Then $q(t) \ge 0$ for $0 \le t \le 1$ and hence inequalities (1.1) and (1.2) are satisfied.*

Proof. Now $I_O(D(t))$ is 1/4-concave by Hadwiger's result. This means that the quantity $q(t)$ defined in equation (11.16) is non-negative for $0 \le t \le 1$. In particular $q(0) \ge 0$ and $q(1) \ge 0$. Now

$$q(1) = \rho_1^3(L_0 - 4(\frac{\pi^3 I_0}{8})^{1/4}) \ , \tag{11.17}$$

and $q(1) > 0$ iff $(L_0/(2\pi))^4 \ge 2I_0/\pi$ which is inequality (1.1) (for domains for which the Steiner centroid coicides with the centroid). Also

$$q(0) = \rho_1(I(\partial D) - 4(\frac{\pi I_0^3}{2})^{1/4}) \ . \tag{11.18}$$

which is inequality (1.2).
Corollary. *For any centrally-symmetric convex set $D = D_0$, inequalities (1.1) and (1.2) are satisfied.*

11.4 Knothe 1957

Some results which appear closely related to inequality (1.2) are suggested by Knothe [15] §3. Knothe proves his results for 3-dimensional domains. To assist readers in adapting these to the 2-dimensions and comparing these results with those in this paper, we record some items concerning the notation in [15]§3:

- O (denoted C in [15]) is the centre of $K = D_0$. Our B is denoted by S in [15].
  Both the centroid and the Steiner centroid are at O.
- Λ is a line through the centre O. ([15] uses a plane P.)
- s is the distance from Λ.
- B is the disk, centre O, with $\int_B s^2 = \int_{D_0} s^2$.

Let us now rotate the coordinate axes so that $s = x$.

Some results in [15] are recognizably the same as some in this paper.
Theorem (Knothe). *Let $D(t) = (1-t)D_0 + tB$, with D_0 and B as above. Write*

$$I_{11}(D) = \int_D x^2$$

so that $I_{11}(D_0) = I_{11}(B)$. Then

$$I_{11}(D(t)) \ge I_{11}(D_0) = I_{11}(B). \tag{11.19}$$

Inequality (11.19) is essentially equation (45) of [15]. This result of Knothe's follows from Theorem 5 (along the lines of the discussion in §7, just before the statement of Lemma 3).

[15] also gives an isoperimetric inequality – in [15] for 3 dimensions, but here adapted to 2 – of the form:

$$\frac{(\int_{\partial D} x^2)^4}{(\int_D x^2)^3} \geq \frac{(\int_{\partial B} x^2)^4}{(\int_B x^2)^3} = \frac{\pi^4}{(\pi/4)^3} = 64\pi \ , \tag{11.20}$$

$$\frac{(\int_{\partial D} y^2)^4}{(\int_D y^2)^3} \geq \frac{(\int_{\partial B} y^2)^4}{(\int_B y^2)^3} = \frac{\pi^4}{(\pi/4)^3} = 64\pi \ . \tag{11.21}$$

See [15], inequality (49). More generally, this is, for any unit direction u.

$$\frac{(\int_{\partial D} \langle x, u \rangle^2)^4}{(\int_D \langle x, u \rangle^2)^3} \geq 64\pi \ .$$

We can re-write our inequality (1.2) as

$$\frac{(\int_{\partial D} r^2)^4}{(\int_D r^2)^3} \geq \frac{(\int_{\partial B} r^2)^4}{(\int_B r^2)^3} = \frac{(2\pi)^4}{(\pi/2)^3} = 128\pi \ . \tag{11.22}$$

We do not know if it is possible to deduce inequality (11.22) from inequalities (11.20,11.21), or vice versa.

Inequality (11.21) is also proved – along with a generalization associated with D lying in a sector rather than merely the special-case of a half-space – in a paper by Payne and Weinberger reported on p3 of [1].

11.5 Further inequalities involving I_0 and $I(\partial D)$

Some little-known isoperimetric inequalities can be easy to find. For example, we have

$$\frac{2I_0}{\pi} \frac{L}{2\pi} \geq \frac{A}{\pi} \frac{I(\partial D)}{2\pi} \ . \tag{11.23}$$

This inequality follows from an application of Chebyshev's inequality ([17], p40). One uses a weight ρ in the integrals and the fact, depending on the positivity of ρ, that p and $p^2 + \dot{p}^2$ vary in the same direction.

12 Conclusion, and further isoperimetric inequalities

There are many related inequalities already in print, including some in [9]. Products of the principal moments are also of interest. Sylvester's problem in geometric probablity was a motivation for some of these. Sylvester's problem is as follows: *Given a convex set D in the plane, and 4 points chosen at random from it, what is the probability that the four points form a convex quadrilateral.* The probablity can be expressed in terms of moments of inertia and the question arose as to which convex sets, with a given area say, make this probability largest, and which smallest. The answers involve ellipses at one extreme and triangles at the other. The key work on this was done by Blaschke and is reported in his 1923 book. (There are many accessible modern references, e.g. [19].)

While the 1/4-concavity of I_c, and its n-dimensional generalization $1/(n+2)$-concavity, was a major result in [9], there were other items in [9] independent of this. These concern products of inertia. To give a flavour of some of these we quote from *Mathematical Reviews*.

> **Busemann's Math. Reviews MR0080942 review of [9]:** A functional $\varphi(K)$ defined on the convex bodies in $\mathbb{R}^n$ is concave if $\varphi(\lambda K+\mu L)\geq\lambda\varphi(K)+\mu\varphi(L)$. The norm of a convex body is its average width multiplied by $\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$. For a concave functional $\varphi(K)$ which is invariant under motion of K the sphere yields the maximal value among all bodies with a given norm. For a given convex body K denote by S and S^0 the spheres which have the same norm and volume as K. Put $I_0(K)=1$ and
>
> $$I_r(K)={c_r}^{-1}\int_K\cdots\int_K|s,p_1,\cdots,p_r|^2dp_1\cdots dp_n(1\leq r\leq n), \tag{12.1}$$
>
> where c_r are numerical constants (given in equation (4c) of [9]), s is the center of gravity of K, $|s,p_1,\cdots,p_r|$ is the volume of the r-simplex with vertices $s,p_1,\cdots,p_r$; and p_j range independently over K. Then
>
> $$I_1(S)\geq I_1(K)\geq I_2(K)^{1/2}\geq\cdots\geq I_n(K)^{1/n}\geq I_n(S^0), \tag{12.2}$$
>
> $$I_a(K)^{b-c}I_b(K)^{c-a}I_c(K)^{a-b}\geq 1 \text{ for } 0\leq a<b<c\leq n. \tag{12.3}$$
>
> $I_1(K)$ is, of course, the ordinary polar moment of inertia of K. The first inequality is proved by showing that $I_1(K)^{1/(n+2)}$ is a concave functional. It was known previously only in the case $n=2$ (see [18] MR 13, 270)

In 2 dimensions, $n=2$, 'having the same norm' means 'having the same perimeter'.

An ingredient in the proof if inequalities (12.2) is the fact that the moment of inertia tensor is positive definite. In particular, all roots of the characteristic polynomial are real, and, as a consequence various coefficient inequalities, given in [10] hold. However, when $n=2$, the chain of inequalities in the inner parts of (12.2) follows simply from the AGM inequality. (Let M be the moment of inertia matrix, with moments taken about the centroid. Let its eigenvalues be λ_1 and λ_2. Then $I_1=\lambda_1+\lambda_2$ and $I_2=4\lambda_1\lambda_2$, and the AGM inequality is that $I_1\geq\sqrt{I_2}$.)

Various contemporary papers are also concerned with ellipses associated with convex domains. A common notation for the Legendre ellipsoid (the ellipsoid associated with the moment of inertia matrix M) is $\Gamma_2 D$. This ellipsoid has the same centroid as D and the same moment of inertia matrix. In 2 dimensions we have the following.

- The moment of inertia matrix for the disk $B(0,\rho_1)$ is $\rho_1^4\pi/4$ times the identity matrix.
- The volume of the Legendre ellipsoid (area in 2 dimensions) is

  $$\text{Volume}(\Gamma_2 D)=2\sqrt{\pi}(\det(M))^{1/4}=2\sqrt{\pi}(\lambda_1\lambda_2)^{1/4},$$

  where λ_1, λ_2 are, as above, the eigenvalues of M.

- The isoperimetric inequality, (when $n = 2$, the rightmost inequality in (12.2)) is given in Blaschke's 1923 book, associated with Sylvester's problem, and is equivalent to the following.

Theorem (Blaschke 1918, John 1973, Petty). *If D is star-shaped about the origin, then $V(\Gamma_2 D) \geq V(D)$ with equality if and only if D is an ellipsoid centred at the origin.*

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G. KEADY,
School of Mathematics and Statistics,
University of Western Australia, 6009, Australia,
email: `keady@maths.uwa.edu.au`