THE STRONG VARIANT OF A BARBASHIN'S THEOREM ON STABILITY OF SOLUTIONS FOR NON-AUTONOMOUS DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. Among others we shall prove that an exponentially bounded evolution family $\mathbf{U} = \{U(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators acting on a Banach space X is uniformly exponentially stable if and only if there exists $q \in [1,\infty)$ such that

$$\sup_{t\geq 0}\left(\int_0^t ||U(t,\tau)^* x^*||^q d\tau\right)^{\frac{1}{q}} = M(x^*) < \infty, \quad \forall x^* \in X^*.$$

This result seems to be new even in the finite dimensional case and it is the strong variant of an old result of E. A. Barbashin ([1]Theorem 5.1).

1. Introduction

The well-known theorem of Datko says that *for an exponentially bounded and strongly continuous evolution family* $\mathbf{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *of bounded linear operators acting on a Banach space* X *the following three statements are equivalent:*

(i) *The family* $\mathbf{U}$ *is uniformly exponentially stable, that is there exist the constants* $\nu > 0$ *and* $N > 0$ *such that*

$$||U(t,s)|| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s \geq 0.$$

(ii) *(Strong Datko's Condition) There exist* $p \geq 1$ *and* $M_p > 0$ *such that*

$$\sup_{s\geq 0}\left(\int_s^\infty ||U(t,s)x||^p dt\right)^{\frac{1}{p}} \leq M_p||x|| \text{ for all } x \in X. \qquad (SDC)$$

(iii) *(Uniform Datko's Condition) The following inequality holds:*

$$\sup_{s\geq 0}\left(\int_s^\infty ||U(t,s)||^p dt\right)^{\frac{1}{p}} := K_p < \infty. \qquad (UDC)$$

The equivalence between **(i)** and **(ii)** can be stated under the general assumption that for each $x \in X$ the map $t \mapsto ||U(t,s)x||$ is measurable even if the family $\mathbf{U}$ is not strongly continuous. It is easily to see that the measurability of the map $t \mapsto ||U(t,s)||$ relies by the strong continuity of the family $\mathbf{U}$. Moreover, either of

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the conditions (SDC) or (UDC) may be replaced by the following more general condition, originally given by S. Rolewicz.

(iv) (Strong Rolewicz's Condition) There exists a nondecreasing function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ which $\phi(t) > 0$ for all $t > 0$ such that

$$\sup_{s\geq 0} \int_s^\infty \phi(||U(t,s)x||)dt < \infty, \qquad \text{for all } x \in X \text{ with } ||x|| \leq 1.$$

Particularly this shows that **(i)** and **(ii)** can be reformulated with $p > 0$ instead of $p \geq 1$.

For much more details, proofs and others formulations of this type theorems we refer to [6], [7], [10], [12], [14],[4], [3] and references therein.

On the other hand a reformulation of an old result of E. A. Barbashin ([[1], Theorem 5.1], [9]) reads as follows:

Let $\mathbf{U} = \{U(t,s) : t \geq s \geq 0\}$ *be an exponentially bounded evolution family of bounded linear operators acting on a Banach space* X *such that for each* $t > 0$ *and each* $x \in X$ *the maps* $s \mapsto ||U(t,s)|| : [0,t] \to \mathbf{R}_+$, *and* $s \mapsto ||U(t,s)x||$ *are measurable. The following statements are equivalent:*

(i) *The family* $\mathbf{U}$ *is uniformly exponentially stable.*

(ii) *(Uniform Barbashin's Condition) There exists* $1 \leq p < \infty$ *such that*

$$\sup_{t\geq 0} \left(\int_s^t ||U(t,s)||^p ds \right)^{\frac{1}{p}} < \infty.$$

A Rolewicz's variant of a similar uniform result for families on the entire real line, can be found in [[3], Theorem 4.1].

It is naturally to ask what is the variant of the Strong Barbashin Condition equivalent with the uniform exponential stability property of the evolution family $\mathbf{U}$? In this paper we shall give an answer to this question.

2. Notations and preliminary results

Let X be a real or complex Banach space and X^* its dual space. By $\mathcal{B}(X)$ will denote the Banach algebra of all linear and bounded operators acting on X. The norms on X, X^* and $\mathcal{B}(X)$ will be denoted by the symbol $||\cdot||$. Let $\mathbf{R}_+ := [0, \infty)$ and J either of $\mathbf{R}$ or $\mathbf{R}_+$. By Δ_J will denote the set of all pairs $(t,s) \in J \times J$ with $t \geq s$, and let $\Delta_J^* := \Delta_J \setminus \{(t,t) : t \geq 0\}$. By *evolution family* of bounded linear operators acting on X will mean a family $\mathbf{U} = \{U(t,s) : (t,s) \in \Delta_J\} \subset \mathcal{B}(X)$ which verifies the following two conditions:

1. $U(t,t) = I$-the identity of $\mathcal{B}(X)$- for all $t \in J$, and

2. $U(t,s)U(s,r) = U(t,r)$ for all $t, s, r \in J$ with $t \geq s \geq r$.

An evolution family is called *strongly continuous* if for each $x \in X$ the maps

$$\tau \mapsto U(\tau,s)x : [s,t] \to X \text{ and } s \mapsto U(t,s)x : [s,t] \to X$$

are continuous for any pair $(t,s) \in \Delta_J$. An evolution family is *exponentially bounded* if there exist $\omega \in \mathbf{R}$ and $M_\omega \geq 1$ such that

$$||U(t,s)|| \leq M_\omega e^{\omega(t-s)} \text{ for all } (t,s) \in \Delta_J. \tag{2.1}$$

If the evolution family $\mathbf{U}$ is exponentially bounded then we may choose a positive ω such that (2.1) holds. An evolution family is *uniformly exponentially stable* if there exists a negative ω such that the relation (2.1) is fulfilled.

Remark 1. *There exist strongly continuous evolution families which is not exponentially bounded. Indeed, let $p:\mathbf{R}\to\mathbf{R}$ the map defined by*

$$p(t)=\begin{cases} m, & \text{if} \quad t\in[m^2,m^2+1) \\ 0, & \text{in the rest}\end{cases}$$

m being an integer number. Let us consider $U(t,s):=\exp(\int_s^t p(\tau)d\tau)$ for $(t,s)\in\Delta_J$ and let $X=\mathbf{R}$. It is clear that $U(t,s)\in\mathcal{B}(\mathbf{R})$ for all $(t,s)\in\Delta_J$ and moreover, the family $\mathbf{U}=\{U(t,s)\}_{t\geq s}$ is a strongly continuous evolution family which is not exponentially bounded. In fact, if suppose that the family $\mathbf{U}$ is exponentially bounded, then

$$U(m^2+1,m^2)=e^m\leq M_\omega e^\omega \text{ for all integer } m,$$

which is a contradiction.

Remark 2. *If an evolution family $\mathbf{U}$ satisfies the convolution condition*

$$U(t,s)=U(t-s,0) \text{ for all } (t,s)\in\Delta_J \tag{2}$$

then the one parameter evolution family $\mathbf{T}:=\{U(t,0),t\geq 0\}$ is a semigroup of operators on X. If $\mathbf{T}$ is strongly continuous then it has exponential growth. The converse statement is not true such as the following example shows.

Example 1. *Let $X=\mathbf{R}$ and $T(t):X\to X$ defined by:*

$$T(t)x=\begin{cases} 0, & \text{if} \quad t>0 \\ x, & \text{if} \quad t=0\end{cases}$$

where $t\geq 0$ and $x\in\mathbf{R}$. It is easily to check that $|T(t)x|\leq|x|$ for every $t\geq 0$ and $x\in X$, so $T(t)\in\mathcal{B}(\mathbf{R})$ and the one parameter family $\mathbf{T}=\{T(t):t\geq 0\}$ is a semigroup of operators on $X=\mathbf{R}$. Moreover, for each $x\in X$, the map $t\mapsto T(t)x$ is continuous on $(0,\infty)$, but for $x\neq 0$ the same map is not continuous at the point $t_0=0$. On the other hand for every $t\geq 0$ we have that $||T(t)||\leq 1$, thus the semigroup $\mathbf{T}$ is exponentially bounded.

An one parameter semigroup $\mathbf{T}$ is called strongly measurable if for each $x\in X$ the map $t\mapsto T(t)x$ is Bochner measurable. See [5] for definitions of different kinds of measurability for vector-valued functions. It is known that every strongly continuous semigroup is also strongly continuous on $(0,\infty)$, but the measurability property does not imply the continuity at the origin such the above example shows. However, a certain type of measurability can be obtained for exponentially bounded semigroups. Precisely, we have:

Proposition 1. *If an one parameter semigroup $\{T(t)\}_{t\geq 0}$ is exponentially bounded then for each $x\in X$, the map $t\mapsto||T(t)x||$ is measurable.*

Proof. Suppose that (2.1) holds. We endow the space X with an equivalent norm given by

$$|||x|||:=\sup_{t\geq 0}||e^{-\omega t}T(t)x||, \quad x\in X.$$

It is clear that the map $x \mapsto |||x|||$ is a norm on X and X endowed with this norm becomes a Banach space. This follows by the inequalities:

$$||x|| \leq |||x||| \leq M_\omega ||x|| \quad x \in X.$$

Let $S(t) := e^{-\omega t}T(t), t \geq 0$. For each positive h, we have:

$$\begin{aligned} |||S(t+h)x||| &= \sup_{s\geq 0} ||e^{-\omega s}T(s)e^{-\omega(t+h)}T(t+h)x|| \\ &= \sup_{r\geq h} ||e^{-\omega r}T(r)(S(t)x)|| \\ &\leq \sup_{r\geq 0} ||e^{-\omega r}T(r)(S(t)x)|| = |||S(t)x|||. \end{aligned}$$

The function $t \mapsto |||S(t)x|||$ is measurable because it is non-increasing, so the function $t \mapsto |||T(t)x||| = e^{\omega t}|||S(t)x|||$ is also measurable. □

Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous one parameter semigroup on a Banach space X and $\mathbf{T}^* = \{T(t)^*\}_{t\geq 0}$ the associated one parameter dual semigroup on X^*. It is known that the dual semigroup $\mathbf{T}^*$ may be not strongly continuous but it is exponentially bounded because $||T(t)|| = ||T(t)^*||$ for all $t \geq 0$. Then for each $x^* \in X^*$ the map $t \mapsto ||T(t)^*x^*||$ is measurable. At this moment we do not know if a similar result for evolution families holds. Throughout in as follows we shall suppose that for each $x \in X$ each $x^* \in X^*$ and each $(t,s) \in \Delta_J$ the maps

$$\tau \mapsto ||U(t,\tau)^*x^*|| : [s,t] \to \mathbf{R}_+ \text{ and } r \mapsto ||U(r,s)x|| : [s,t] \to \mathbf{R}_+$$

are measurable.

3. Uniform Stability

Let $p \in (1,\infty)$ and $q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We are in the position to state the our first result on uniform stability of evolution families.

Theorem 1. *Let $\mathbf{U} = \{U(t,s) : (t,s) \in \Delta_J\}$ be an evolution families which verifies the measurability conditions stated above. The following two conditions are equivalent:*

(i) *The evolution family is uniformly stable, that is*

$$\sup_{(t,s)\in\Delta_J} ||U(t,s)|| = M_J < \infty. \tag{3.1}$$

(ii) *There exist the positive constants M_p and M_q^* such that*

$$\sup_{(t,s)\in\Delta_J^*} \left(\frac{1}{t-s} \int_s^t ||U(t,\tau)^*x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^* \tag{3.2}$$

and

$$\sup_{(t,s)\in\Delta_J^*} \left(\frac{1}{t-s} \int_s^t ||U(\tau,s)x||^p d\tau \right)^{\frac{1}{p}} \leq M_p ||x||, \quad \forall x \in X. \tag{3.3}$$

Proof. The implication (**i**) $\Rightarrow$ (**ii**) is trivial having in mind that $||U(t,s)|| = ||U(t,s)^*||$. In this case can choose $M_p = M_q^* = M_J$. Now we are stating the proof of (**ii**) $\Rightarrow$ (**i**). Let $x \in X, x^* \in X^*$ and $(t,s) \in \Delta_J^*$. Using the Hölder inequality we get:

$$(t-s)|\langle x^*, U(t,s)x\rangle| = \int_s^t |\langle x^*, U(t,\tau)U(\tau,s)x\rangle| d\tau$$

$$\leq \left(\frac{1}{t-s}\int_s^t |\langle ||U(t,\tau)^*x^*||^q d\tau\right)^{\frac{1}{q}} \cdot \left(\frac{1}{t-s}\int_s^t ||U(\tau,s)x||^p d\tau\right)^{\frac{1}{p}} (t-s)$$

$$\leq (t-s)M_pM_q^*||x||\cdot||x^*||.$$

Finally we get:

$$\sup_{(t,s)\in\Delta_J} ||U(t,s)|| = \sup_{||x||\leq 1, ||x^*||\leq 1} |\langle x^*, U(t,s)x\rangle| \leq \max\{1, M_p, M_q^*\} := M_J < \infty,$$

that is (3.1) holds.

□

Corollary 1. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be an exponentially bounded one parameter semigroup on a Banach space* X. *The following two statements are equivalent:*

(i) *The semigroup* $\mathbf{T}$ *is uniformly stable (or bounded), that is*

$$\sup_{t\geq 0} ||T(t)|| = M < \infty.$$

(ii) *There exist* $p, q \in (1,\infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and the positive constants* M_p *and* M_q^* *such that*

$$\sup_{t>0}\left(\frac{1}{t}\int_0^t ||T(\tau)x||^p d\tau\right)^{\frac{1}{q}} \leq M||x||, \quad \forall x \in X \tag{3.4}$$

and

$$\sup_{t>0}\left(\frac{1}{t}\int_0^t ||T(\tau)^*x^*|| d\tau\right)^{\frac{1}{q}} \leq M_q^*||x^*||, \quad \forall x^* \in X^*. \tag{3.5}$$

Proof. The measurability of the maps $\tau \mapsto ||T(\tau)x||$ and $\tau \mapsto ||T(\tau)^*x^*||$ relies by the fact that the semigroup $\mathbf{T}$ is exponentially bounded as been stated in the above Proposition 1. See also the comments after its proof. The inequalities (3.4) and (3.5) can be easily obtained from (3.2) and respectively from (3.3) by making changes of variables in integrals and using the convolution condition (2.2). □

Remark 3. *The result from Corollary 1 in the particular case of strongly continuous semigroups was announced earlier by Hans Zwart (*[16]*).*

Let $1 \leq p < \infty$. A strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ on a Hilbert space $H$is called *weakly-L^p-stable* if

$$\int_0^\infty |\langle T(t)x, y\rangle| dt < \infty \quad \text{for all } x, y \in H.$$

It is already well-known that every strongly continuous semigroup is weakly-L^p-stable if and only if it is uniformly exponentially stable. For details and proofs we refer to [11], [8], [15], [14]. See also the references therein. Having in mind the result from the above Corollary 1 it is naturally to ask if a strongly continuous semigroup on a Hilbert space H which verifies the condition:

$$\sup_{t>0} \frac{1}{t} \int_0^t |\langle T(\tau)x, y\rangle|^2 d\tau < \infty \quad \text{for all } x, y \in H,$$

is bounded. We do not give here an answer to this question.

Corollary 2. *An exponentially bounded one parameter semigroup $\{T(t)\}$ on a Banach space X for which the map $t \mapsto ||T(t)||$ is measurable, is uniformly bounded, if and only if*

$$\sup_{t>0} \frac{1}{t} \int_0^t ||T(\tau)||^2 d\tau < \infty.$$

Corollary 3. *An one parameter group $\{G(t)\}_{t\in\mathbf{R}}$ on a Banach space X for which the map $t \mapsto ||G(t)||$ is measurable, is uniformly bounded on $\mathbf{R}_+$, if and only if*

$$\sup_{t>0} \frac{1}{t} \int_0^t ||G(\tau)||^2 d\tau < \infty.$$

Proof. The map $t \mapsto g(t) := \ln(||G(t)||) : \mathbf{R} \to [-\infty, \infty)$ is measurable and subadditive, that is $g(t+s) \le g(t) + g(s)$ for all $t, s \in \mathbf{R}$. Thus it is superior locally bounded. Then the restriction of G to $\mathbf{R}_+$ is locally bounded or equivalently the semigroup $\mathbf{G}_+ = \{G(t) : t \ge 0\}$ is exponentially bounded. Then we can apply the above Corollary 2 in order to finish the proof. □

Remark 4. *The implication* (**ii**) ⇒ (**i**) *from Corollary 1 cannot be preserved if either of the relations (3.4) or (3.5) is removed such that the following example from* [13] *shows.*

Example 2. *Let $X = L^2(\mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}_+$ the function given by $g(s) = \sqrt{(1+|s|)}$. For each $t \ge 0$ and each $f \in X$ let us consider the bounded linear operator $T(t)$ defined by:*

$$(T(t)f)(s) = \frac{g(t+s)}{g(s)} f(t+s), \quad s \in \mathbf{R}, \quad t \ge 0.$$

It is easily to check that the family $\mathbf{T} = \{T(t)\}_{t\ge 0}$ is a strongly continuous semigroup of bounded linear operators on X and $||T(t)|| = \sqrt{(1+t)}$, that is $\mathbf{T}$ is not uniformly stable. However, the relation (3.4), with $p = 2$, is verified by $\mathbf{T}$.

4. Uniform exponential stability

In order to prove the first result of this section we need a Lemma which reads as follows:

Lemma 1. *Let $\mathbf{U} = \{U(t,s), (t,s) \in \Delta_J\}$ be an exponentially bounded evolution family of bounded linear operators on a Banach space X. If there exists a function $g : \mathbf{R}_+ \to \mathbf{R}_+$ such that*

$$\inf_{t>0} g(t) < 1 \text{ and } ||U(t,s)|| \le g(t-s), \text{ for all } t \ge s \in J,$$

then the family $\mathbf{U}$ is uniformly exponentially stable.

For the proof of the above Lemma we refer to [[2], Lemma 4].

Theorem 2. *Let* $\mathbf{U} = \{U(t,s) : (t,s) \in \Delta_J\}$ *be an exponentially bounded evolution family on a Banach space* X. *The following statements are equivalent:*

(i) *The family* $\mathbf{U}$ *is uniformly exponentially stable.*

(ii) *There exist* $p, q \in (1,\infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and the positive constants* M_p *and* M_q^* *such that*

$$\sup_{t\in\mathbf{R}} \left(\int_{-\infty}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^* \text{ if } J = \mathbf{R} \tag{4.1}$$

or

$$\sup_{t\geq 0} \left(\int_{0}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^* \text{ if } J = \mathbf{R}_+ \tag{4.2}$$

and

$$\sup_{(t,s)\in\Delta_J^*} \left(\frac{1}{t-s} \int_{s}^{t} ||U(\tau,s)x||^p d\tau \right)^{\frac{1}{p}} \leq M_p ||x||, \quad \forall x \in X.$$

Proof. The implication (**i**) $\Rightarrow$ (**ii**) is trivial. In order to prove the converse implication we remark first that if (4.1) holds then the inequality

$$\sup_{(t,s)\in\Delta_J^*} \left(\frac{1}{t-s} \int_{s}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^* \text{ if } J = \mathbf{R}_+$$

holds as well. Indeed if $t - s \geq 1$ then

$$\left(\frac{1}{t-s} \int_{s}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq \left(\int_{s}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}}$$

$$\leq \left(\int_{-\infty \text{ or } 0}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^*.$$

If $0 < t - s < 1$ then we have:

$$\left(\frac{1}{t-s} \int_{s}^{t} ||U(t,\tau)^* x^*||^q) d\tau \right)^{\frac{1}{q}} \leq \left(\frac{1}{t-s} \int_{s}^{t} M_\omega^q e^{\omega q} ||x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_\omega e^{\omega} ||x^*||$$

where ω was been considered positive. Thus as stated in the above Theorem 1, the evolution family $\mathbf{U}$ is uniformly bounded, that is there exists a positive constant N_1 such that

$$||U(t,s)|| \leq N_1 \text{ for all } (t,s) \in \Delta_J. \tag{4.3}$$

On the other hand for each $(t,s)\in\Delta_J^*$ we have:

$$\begin{aligned}
(t-s)|\langle x^*, U(t,s)x\rangle| &= \int_s^t |\langle x^*, U(t,\tau)U(\tau,s)x\rangle| d\tau \\
&\leq \left(\int_s^t ||U(t,\tau)^*x^*||^q d\tau\right)^{\frac{1}{q}} \left(\int_s^t ||U(\tau,s)x||^p d\tau\right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{t-s}\int_s^t ||U(\tau,s)x||^p d\tau\right)^{\frac{1}{p}} \left(\int_s^t ||U(t,\tau)^*x^*||^q d\tau\right)^{\frac{1}{q}} (t-s)^{\frac{1}{p}} \\
&\leq M_p||x||(t-s)^{\frac{1}{p}} \left(\int_{-\infty \text{ or } 0}^t ||U(t,\tau)^*x^*||^q d\tau\right)^{\frac{1}{q}} \\
&\leq M_p M_q^* ||x||\cdot||x^*||(t-s)^{\frac{1}{p}}.
\end{aligned}$$

As a consequence there exists a positive constant N_2 such that

$$(t-s)^{\frac{1}{q}}||U(t,s)|| \leq N_2, \quad \forall (t,s)\in\Delta_J. \tag{4.4}$$

If add the inequalities (4.3) and (4.4) we get the following estimation for the norm of $U(t,s)$:

$$||U(t,s)|| \leq \frac{N_1+N_2}{1+(t-s)^{\frac{1}{q}}}.$$

Then we can use the above Lemma 1 in order to finish the proof. □

Corollary 4. *An uniformly bounded evolution family $\mathbf{U}=\{U(t,s):(t,s)\in\Delta_J\}$ is uniformly exponentially stable if and only if it verifies the conditions (4.2) (in the case $J=\mathbf{R}_+$) or verifies the condition (4.1) (in the case $J=\mathbf{R}$), from the above Theorem 2.*

The following theorem may be obtained by the duality principle.

Theorem 3. *Let $\mathbf{U}=\{U(t,s):(t,s)\in\Delta_J\}$ be an exponentially bounded evolution family on a Banach space X. The following statements are equivalent:*

(i) *The family $\mathbf{U}$ is uniformly exponentially stable.*

(ii) *There exist $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$ and the positive constants M_p and M_q^* such that*

$$\sup_{(t,s)\in\Delta_J}\left(\int_s^t ||U(\tau,s)x||^p d\tau\right)^{\frac{1}{p}} \leq M_p||x||, \quad \forall x\in X \tag{4.5}$$

and

$$\sup_{(t,s)\in\Delta_J^*}\left(\frac{1}{t-s}\int_s^t ||U(t,\tau)^*x^*||^q d\tau\right)^{\frac{1}{q}} \leq M_q^*||x^*||, \quad \forall x^*\in X^*. \tag{4.6}$$

It is clear that the relation (4.5) is equivalent with the Strong Datko Condition (SDC) and then the condition (4.6) from the above Theorem 3 may be removed.

The following result shows that the uniform bounded-ness assumption from the above Corollary 4 can be also replaced with a more generally one, namely with the exponentially bounded-ness assumption. Precisely we can state the following result:

Theorem 4. *Let $\mathbf{U} = \{U(t,s), (t,s) \in \Delta_J\}$ be an exponentially bounded evolution family on a Banach space X which verifies the measurability conditions stated in the end of the second section of our note. The following two statements are equivalent:*

(i) *The family $\mathbf{U}$ is uniformly exponentially stable.*

(ii) *There exist $q \in [1, \infty)$ and a positive constant M_q^* such that*

$$\sup_{t\in\mathbf{R}} \left(\int_{-\infty}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^* \text{ if } J = \mathbf{R}$$

or

$$\sup_{t\geq 0} \left(\int_{0}^{t} ||U(t,\tau)^* x^*||^q d\tau \right)^{\frac{1}{q}} \leq M_q^* ||x^*||, \quad \forall x^* \in X^* \text{ if } J = \mathbf{R}_+.$$

Proof. Let $s \in J$ be fixed. Then for $t \geq s+1$ we have that

$$\begin{aligned} &\int_0^1 M_\omega^{-q} e^{-\omega q u} du |\langle x^*, U(t,s)x\rangle|^q \\ &\leq \left(\int_s^t ||U(t,\tau)^* x^*||^q \cdot ||U(t,s)||^q M_\omega^{-q} e^{-\omega(\tau - s)q} d\tau \right) ||x||^q \\ &\leq \left(\int_s^t ||U(t,\tau)^* x^*||^q d\tau \right) \cdot ||x||^q \leq (M_q^*)^q ||x^*||^q ||x||^q \end{aligned}$$

while that for $s \leq t < s+1$ we get $||U(t,s|| \leq M_\omega e^\omega$. Thus the family $\mathbf{U}$ is uniformly bounded, and then can apply the above Corollary 4 in order to finish the proof in the case $q \in (1, \infty)$. In the case $q = 1$ and $J = \mathbf{R}_+$ we have:

$$(t-s)|\langle x^*, U(t,s)x\rangle| \leq M_{\mathbf{R}_+} \int_0^t ||U(t,\tau)^* x^*|| d\tau ||x|| \text{ for all } (t,s) \in \Delta_{\mathbf{R}_+}.$$

Similar estimation can be easily established in the case $J = \mathbf{R}$. Finally we apply again the above Lemma 1 in order to finish the proof.

□

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