

# A COMPLETELY MONOTONIC FUNCTION INVOLVING DIVIDED DIFFERENCES OF PSI AND POLYGAMMA FUNCTIONS AND AN APPLICATION

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ABSTRACT. A function involving the divided differences of the psi function and the polygamma functions is proved to be completely monotonic. As an application of this result, the monotonicity and convexity of a function originated from establishing the best upper and lower bounds in Kershaw's inequality is deduced.

## 1. INTRODUCTION

Recall [5] that a function  $f$  is said to be completely monotonic on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  and  $(-1)^n f^{(n)}(x) \geq 0$  for  $x \in I$  and  $n \geq 0$ . For information about the history, applications and recent developments on the completely monotonic function, please refer to the expository article [5] and the references therein.

The Kershaw's inequality [4] states that

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \quad (1)$$

for  $0 < s < 1$  and  $x \geq 1$ , where  $\Gamma$  denotes the classical Euler's gamma function and the middle term in (1) is a special case of the Wallis' function  $\frac{\Gamma(x+p)}{\Gamma(x+q)}$  for  $x+p > 0$  and  $x+q > 0$ . It is clear that inequality (1) can be rearranged as

$$\frac{s}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} - x < \sqrt{s + \frac{1}{4}} - \frac{1}{2}. \quad (2)$$

Let  $s$  and  $t$  be nonnegative numbers and  $\alpha = \min\{s, t\}$ . Define

$$z_{s,t}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t \\ e^{\psi(x+s)} - x, & s = t \end{cases} \quad (3)$$

in  $x \in (-\alpha, \infty)$ . Standard differentiating and simplifying yields

$$z'_{s,t}(x) = [z_{s,t}(x) + x] \frac{\psi(x+t) - \psi(x+s)}{t-s} - 1, \quad (4)$$

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$$z''_{s,t}(x) = [z_{s,t}(x) + x] \left\{ \left[ \frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s} \right\} \quad (5)$$

$$= \frac{z_{s,t}(x) + x}{(t-s)^2} \left\{ [\psi(x+t) - \psi(x+s)]^2 + (t-s)[\psi'(x+t) - \psi'(x+s)] \right\}. \quad (6)$$

In order to obtain the best upper and lower bounds for double inequality (1) or (2), the monotonicity and convexity properties of the function  $z_{s,t}(x)$  in  $x \in (-\alpha, \infty)$  is showed in [2, 3, 7] by using Laplace transform and other complicated techniques respectively.

Let

$$\Theta_{s,t}(x) = [\psi(x+t) - \psi(x+s)]^2 + (t-s)[\psi'(x+t) - \psi'(x+s)] \quad (7)$$

and

$$\Delta_{s,t}(x) = \begin{cases} \left[ \frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s \neq t \\ [\psi'(x+s)]^2 + \psi''(x+s), & s = t \end{cases} \quad (8)$$

in  $x \in (-\alpha, \infty)$ . It is clear from (5) and (6) that

$$z''_{s,t}(x) = [z_{s,t}(x) + x]\Delta_{s,t}(x) = \frac{z_{s,t}(x) + x}{(t-s)^2}\Theta_{s,t}(x) \quad (9)$$

for  $t \neq s$ .

The aim of this paper is to prove the completely monotonic property of the functions  $\Theta_{s,t}(x)$  and  $\Delta_{s,t}(x)$  in  $(-\alpha, \infty)$ .

**Theorem 1.** *The functions  $\Theta_{s,t}(x)$  for  $|t-s| < 1$  and  $-\Theta_{s,t}(x)$  for  $|t-s| > 1$  are completely monotonic in  $(-\alpha, \infty)$ . The functions  $\Delta_{s,t}(x)$  for  $|t-s| < 1$  and  $-\Delta_{s,t}(x)$  for  $|t-s| > 1$  are completely monotonic in  $x \in (-\alpha, \infty)$ .*

*Remark 1.* Note that, among other things, the positivity of the function  $\Delta_{0,0}(x) = [\psi'(x)]^2 + \psi''(x)$  in (8) has been verified in [1].

As a straightforward application of Theorem 1, the monotonicity and convexity of the function  $z_{s,t}(x)$  is obtained.

**Theorem 2** ([2, 3, 7]). *The function  $z_{s,t}(x)$  in  $(-\alpha, \infty)$  is either convex and decreasing for  $|t-s| < 1$  or concave and increasing for  $|t-s| > 1$ .*

## 2. PROOFS OF THEOREMS

The basic tool of this paper is the following lemma.

**Lemma 1.** *Let  $f(x)$  be defined in an infinite interval  $I$ . If  $\lim_{x \rightarrow \infty} f(x) = 0$  and  $f(x) - f(x+\varepsilon) > 0$  for any given  $\varepsilon > 0$ , then  $f(x) > 0$  in  $I$ .*

*Proof.* By induction, for any  $x \in I$ , we have

$$f(x) > f(x+\varepsilon) > f(x+2\varepsilon) > \cdots > f(x+k\varepsilon) \rightarrow 0$$

as  $k \rightarrow \infty$ . The proof of Lemma 1 is complete.  $\square$

**2.1. Proof of Theorem 1.** It is well known that for any positive integer  $n \in \mathbb{N}$  the psi function  $\psi(x)$  and the polygamma or multigamma functions  $\psi^{(n)}(x)$  have the following integral expressions

$$\psi(x) = \ln x + \int_0^\infty \left[ \frac{1}{u} - \frac{1}{1 - e^{-u}} \right] e^{-xu} \, du \quad (10)$$

and

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1 - e^{-u}} e^{-xu} \, du. \quad (11)$$

Using

$$\psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i} \quad (12)$$

for  $i \in \mathbb{N}$  and  $x > 0$  and direct computing gives

$$\begin{aligned} \Theta_{s,t}(x) - \Theta_{s,t}(x+1) &= \{[\psi(x+t) + \psi(x+t+1)] - [\psi(x+s) + \psi(x+s+1)]\} \\ &\quad \times \{[\psi(x+t) - \psi(x+t+1)] - [\psi(x+s) - \psi(x+s+1)]\} \\ &\quad + (t-s) \{[\psi'(x+t) - \psi'(x+t+1)] - [\psi'(x+s) - \psi'(x+s+1)]\} \\ &= \left\{ \frac{[\psi(x+t+1) + \psi(x+t)] - [\psi(x+s+1) + \psi(x+s)]}{t-s} \right. \\ &\quad \left. - \frac{2x+s+t}{(x+s)(x+t)} \right\} \frac{(t-s)^2}{(x+s)(x+t)} \triangleq \Lambda_{s,t}(x) \frac{(t-s)^2}{(x+s)(x+t)} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \Lambda_{s,t}(x) - \Lambda_{s,t}(x+1) &= \frac{1}{t-s} \left( \frac{1}{x+s} + \frac{1}{x+s+1} - \frac{1}{x+t} - \frac{1}{x+t+1} \right) \\ &\quad - \frac{2x^2 + 2(s+t+1)x + s^2 + t^2 + s + t}{(x+s)(x+s+1)(x+t)(x+t+1)} \\ &= \frac{1 - (s-t)^2}{(x+s)(x+s+1)(x+t)(x+t+1)}. \end{aligned}$$

Since  $\lim_{x \rightarrow \infty} \Lambda_{s,t}^{(i)}(x) = 0$  for any nonnegative integer  $i$  by (10) and (11), and the function  $\frac{\Lambda_{s,t}(x) - \Lambda_{s,t}(x+1)}{1 - (s-t)^2}$  is completely monotonic, that is,

$$(-1)^i \frac{[\Lambda_{s,t}(x) - \Lambda_{s,t}(x+1)]^{(i)}}{1 - (s-t)^2} = \frac{(-1)^i \Lambda_{s,t}^{(i)}(x) - (-1)^i \Lambda_{s,t}^{(i)}(x+1)}{1 - (s-t)^2} \geq 0,$$

in  $(-\alpha, \infty)$ , then  $\frac{(-1)^i \Lambda_{s,t}^{(i)}(x)}{1 - (s-t)^2} \geq 0$  follows from Lemma 1. This means the function  $\frac{\Lambda_{s,t}(x)}{1 - (s-t)^2}$  is completely monotonic in  $(-\alpha, \infty)$ .

Since the function  $\frac{(t-s)^2}{(x+s)(x+t)}$  is completely monotonic and a product of two completely monotonic functions is also completely monotonic, then the function  $\frac{\Theta_{s,t}(x) - \Theta_{s,t}(x+1)}{1 - (s-t)^2}$  is completely monotonic in  $(-\alpha, \infty)$  by considering (13), which is equivalent to

$$(-1)^k \left[ \frac{\Theta_{s,t}(x) - \Theta_{s,t}(x+1)}{1 - (s-t)^2} \right]^{(k)} = \frac{(-1)^k \Theta_{s,t}^{(k)}(x) - (-1)^k \Theta_{s,t}^{(k)}(x+1)}{1 - (s-t)^2} \geq 0$$

for nonnegative integer  $k$ . Further, from  $\lim_{x \rightarrow \infty} \Theta_{s,t}^{(k)}(x) = 0$  for nonnegative integer  $k$ , which can be deduced by utilizing (10) and (11), and Lemma 1, it is concluded

that  $\frac{(-1)^k \Theta_{s,t}^{(k)}(x)}{1-(s-t)^2} \geq 0$  for any nonnegative integer  $k$ . This implies  $(-1)^k \Theta_{s,t}^{(k)}(x) \geq 0$  if and only if  $|t-s| \leq 1$ . Therefore, the functions  $\Theta_{s,t}(x)$  for  $|t-s| < 1$  and  $-\Theta_{s,t}(x)$  for  $|t-s| > 1$  are completely monotonic in  $(-\alpha, \infty)$ .

Since  $\Theta_{s,t}(x) = (t-s)^2 \Delta_{s,t}(x)$ , the function  $\Delta_{s,t}(x)$  has the same monotonicity property as  $\Theta_{s,t}(x)$  in  $(-\alpha, \infty)$ . The proof of Theorem 1 is complete.

**2.2. Proof of Theorem 2.** By Theorem 1, it is easy to see that  $\Theta_{s,t}(x) \geq 0$  and  $\Delta_{s,t}(x) \geq 0$  in  $(-\alpha, \infty)$  if and only if  $|t-s| \leq 1$ . Then  $z''_{s,t}(x) \geq 0$  for  $|t-s| \leq 1$  follows from formula (9). The convexity and concavity of the function  $z_{s,t}(x)$  is proved.

In [6], the inequality

$$\exp[(s-r)\psi(s)] > \frac{\Gamma(s)}{\Gamma(r)} > \exp[(s-r)\psi(r)] \quad (14)$$

for  $s > r > 0$  was obtained, which is equivalent to

$$\max\{e^{\psi(s)}, e^{\psi(r)}\} > \left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)} > \min\{e^{\psi(s)}, e^{\psi(r)}\}$$

for any positive numbers  $s > 0$  and  $t > 0$ . This implies

$$\begin{aligned} z'_{s,t}(x) &= \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \frac{\psi(x+t) - \psi(x+s)}{t-s} - 1 \\ &< e^{\psi(x+t)} \frac{\psi(x+t) - \psi(x+s)}{t-s} - 1 \\ &= e^{\psi(x+t)} \psi'(x+\xi) - 1 < \psi'(x+t) e^{\psi(x+t)} - 1 \end{aligned} \quad (15)$$

and

$$\begin{aligned} z'_{s,t}(x) &> e^{\psi(x+s)} \frac{\psi(x+t) - \psi(x+s)}{t-s} - 1 \\ &= e^{\psi(x+s)} \psi'(x+\xi) - 1 \\ &> \psi'(x+s) e^{\psi(x+s)} - 1, \end{aligned} \quad (16)$$

if assuming  $t > s > 0$  without loss of generality, where  $\xi \in (s, t)$ .

By inequality

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \quad (17)$$

for  $x > 0$ , we obtain

$$x\psi'(x)e^{-1/x} < \psi'(x)e^{\psi(x)} < x\psi'(x)e^{-1/2x} \quad (18)$$

for  $x > 0$ . Using the asymptotic representation

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \dots \quad (19)$$

as  $x \rightarrow \infty$  yields

$$\lim_{x \rightarrow \infty} [x\psi'(x)e^{-1/x}] = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} [x\psi'(x)e^{-1/2x}] = 1. \quad (20)$$

Hence,

$$\lim_{x \rightarrow \infty} [\psi'(x)e^{\psi(x)}] = 1. \quad (21)$$

Combining (21) with (15) and (16) leads to

$$\lim_{x \rightarrow \infty} z'_{s,t}(x) \leq \lim_{x \rightarrow \infty} [\psi'(x+t)e^{\psi(x+t)}] - 1 = \lim_{x+t \rightarrow \infty} [\psi'(x+t)e^{\psi(x+t)}] - 1 = 0$$

and

$$\lim_{x \rightarrow \infty} z'_{s,t}(x) \geq \lim_{x \rightarrow \infty} [\psi'(x+s)e^{\psi(x+s)}] - 1 = \lim_{x+s \rightarrow \infty} [\psi'(x+s)e^{\psi(x+s)}] - 1 = 0.$$

Thus, it is concluded that  $\lim_{x \rightarrow \infty} z'_{s,t}(x) = 0$ .

Since  $z''_{s,t}(x) \gtrless 0$  in  $x \in (-\alpha, \infty)$  for  $|t-s| \leq 1$ , then the function  $z'_{s,t}(x)$  is increasing/decreasing in  $x \in (-\alpha, \infty)$  for  $|t-s| \leq 1$ . Thus, it follows that  $z'_{s,t}(x) \lesseqgtr 0$  and  $z_{s,t}(x)$  is decreasing/increasing in  $x \in (-\alpha, \infty)$  for  $|t-s| \leq 1$ . The monotonicity of the function  $z_{s,t}(x)$  is proved.

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