A CLASS OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS AND APPLICATION TO THE BEST BOUNDS IN THE SECOND GAUTSCHI-KERSHAW'S INEQUALITY

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ABSTRACT. In this article, the logarithmically complete monotonicity of the function $[\Gamma(x+b)/\Gamma(x+a)]^{1/(a-b)}\exp[\psi(x+c)]$ are discussed, where a,b,c are real numbers and Γ is the classical Euler's gamma function. From this, the best upper and lower bounds for Walls' ratio $\Gamma(x+1)/\Gamma(x+s)$ are established, which refine the second Gautschi-Kershaw's inequality.

1. INTRODUCTION

It is well known that the classical Euler's gamma function Γ can be defined for $x>0$ as $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}\,t$. The digamma or psi function ψ is defined as the logarithmic derivative of Γ and $\psi^{(i)}$ for $i\in\mathbb{N}$ are called polygamma functions.

Recall [26] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x)\ge 0$ for $x\in I$ and $n\ge 0$. Recall [2, 13, 20, 21, 22, 23] also that a function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0\le(-1)^k[\ln f(x)]^{(k)}<\infty$ for all $k\in\mathbb{N}$ on I. For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I are denoted respectively by $\mathcal{C}[I]$ and $\mathcal{L}[I]$. In [2, 20, 21], it has been proved that $\mathcal{L}[I]\subset\mathcal{C}[I]$. For more information on the classes $\mathcal{C}[I]$ and $\mathcal{L}[I]$, please refer to [2, 13, 20, 21, 22, 23] and the references therein.

The first and second Gautschi-Kershaw inequalities [5, 7, 9, 24] state that

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

and

$$\exp\left[(1-s)\psi\big(x+\sqrt{s}\,\big)\right]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{2}$$

for $s\in(0,1)$ and $x\ge 1$.

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In [15], among other things, the increasing monotonicity of $[\Gamma(s)/\Gamma(r)]^{1/(s-r)}$ for $s>0$ and $r>0$ and inequality

$$\exp[(s-r)\psi(s)] > \frac{\Gamma(s)}{\Gamma(r)} > \exp[(s-r)\psi(r)] \tag{3}$$

for $s>r>0$ were obtained.

Inequalities (1), (2) and (3) give the lower and upper bounds for the well known Wallis' ratio $\Gamma(x+1)/\Gamma(x+s)$.

In [3], it was proved that the functions

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\left(x+\frac{s}{2}\right)^{s-1} \in \mathcal{C}[(0,\infty)] \tag{4}$$

and

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \in \mathcal{C}[(0,\infty)] \tag{5}$$

for $s\in(0,1)$.

Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. In [5, Theorem 5] and [24], the result (5) was generalized to

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)}\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] \in \mathcal{L}[(-\alpha,\infty)]. \tag{6}$$

In [14, 18], the monotonicity of the function

$$q_{\alpha,\beta}(t)=\begin{cases}\dfrac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t\ne 0\\ \beta-\alpha, & t=0\end{cases} \tag{7}$$

for real numbers α and β with $(\alpha,\beta)\notin\{(0,1),(1,0)\}$ and $\alpha\ne\beta$ was established, and then the paper [10] considered the logarithmically complete monotonicity of the more general function

$$H_{a,b,c}(x)=(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{8}$$

for $x\in(-\rho,\infty)$, where a, b and c are real numbers and $\rho=\min\{a,b,c\}$, and obtained the following conclusions:

(1) $H_{a,b,c}(x)\in\mathcal{L}[(-\rho,\infty)]$ if $(a,b,c)\in\left\{a+b\ge 1,\ c\le b<c+\frac{1}{2}\right\}\cup\left\{a>b\ge c+\frac{1}{2}\right\}\cup\{2a+1\le a+b\le 1,\ a<c\}\cup\{b-1\le a<b\le c\}\setminus\{a=c+1,\ b=c\}$.

(2) $[H_{a,b,c}(x)]^{-1}\in\mathcal{L}[(-\rho,\infty)]$ if $(a,b,c)\in\left\{(a,b,c):a+b\ge 1,\ c\le a<c+\frac{1}{2}\right\}\cup\left\{(a,b,c):b>a\ge c+\frac{1}{2}\right\}\cup\{(a,b,c):b<a\le c\}\cup\{(a,b,c):b+1\le a,\ c\le a\le c+1\}\cup\{(a,b,c):b+c+1\le a+b\le 1\}\setminus\{(a,b,c):a=c+1,\ b=c\}\setminus\{(a,b,c):b=c+1,\ a=c\}$.

These (logarithmically) complete monotonicity mentioned above can be applied to acquire the best bounds in Gautschi-Kershaw's inequalities (1) and (2). For more detailed information, please refer to [4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 19, 24, 25] and the references therein.

The main aim of this article is to generalize the logarithmically complete monotonicity (6). The main result of this paper is the following Theorem 1.

Theorem 1. *Let a, b and c be real numbers and $\rho = \min\{a,b,c\}$. Define*

$$F_{a,b,c}(x) = \begin{cases} \left[\dfrac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(a-b)} \exp[\psi(x+c)], & a \ne b \\ \exp[\psi(x+c) - \psi(x+a)], & a = b \ne c \end{cases} \tag{9}$$

for $x \in (-\rho, \infty)$. Furthermore, let $\theta(t)$ be an implicit function defined by equation

$$e^t - t = e^{\theta(t)} - \theta(t) \tag{10}$$

in $(-\infty, \infty)$. Then $\theta(t)$ is decreasing and $t\theta(t) < 0$ for $\theta(t) \ne t$, and

(1) *$F_{a,b,c}(x) \in \mathcal{L}[(-\rho, \infty)]$ if $(a,b,c) \in D_1(a,b,c)$, where*

$$D_1(a,b,c) = \{c \ge a, c \ge b\} \cup \{c \ge a, 0 \ge c - b \ge \theta(c-a)\} \\ \cup \{c \le a, c - b \ge \theta(c-a)\} \setminus \{a = b = c\}; \tag{11}$$

(2) *$[F_{a,b,c}(x)]^{-1} \in \mathcal{L}[(-\rho, \infty)]$ if $(a,b,c) \in D_2(a,b,c)$, where*

$$D_2(a,b,c) = \{c \le a, c \le b\} \cup \{c \ge a, c - b \le \theta(c-a)\} \\ \cup \{c \le a, 0 \le c - b \le \theta(c-a)\} \setminus \{a = b = c\}. \tag{12}$$

Remark 1. The numerical computation of $\theta(t)$ defined by (10) can be carried out by using the well known software MATHEMATICA 5.2, for example, as follows:

$$\begin{aligned} &\theta(0.5) = -0.599\cdots, && \theta(1) = -1.4937\cdots, && \theta(1.5) = -2.928\cdots, \\ &\theta(2) = -5.3844\cdots, && \theta(2.5) = -9.6824\cdots, && \theta(3) = -17.085\cdots, \\ &\theta(-0.5) = 0.42864\cdots, && \theta(-1) = 0.75078\cdots, && \theta(-1.5) = 1.0028\cdots, \\ &\theta(-2) = 1.2065\cdots, && \theta(-2.5) = 1.3756\cdots, && \theta(-3) = 1.5193\cdots. \end{aligned}$$

As an application of Theorem 1, the following inequalities are obtained.

Theorem 2. *Let $D_1(a,b,c)$ and $D_2(a,b,c)$ be defined by (11) and (12) respectively.*

If $(a,b,c) \in D_1(a,b,c)$, then

$$\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} < \exp[\psi(x+c)] \tag{13}$$

for $x \in (-\rho, \infty)$ and

$$\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} \ge \left[\frac{\Gamma(\delta+b)}{\Gamma(\delta+a)}\right]^{1/(b-a)} \exp[\psi(x+c) - \psi(\delta+c)] \tag{14}$$

for $x \in [\delta, \infty)$ are valid, where δ is a constant greater than $-\rho$.

If $(a,b,c) \in D_2(a,b,c)$, inequalities (13) and (14) are reversed.

As a direct consequence of Theorem 2, the best lower and upper bounds for Wallis's ratio $\Gamma(x+1)/\Gamma(x+s)$ are established below, which improve the second Gautschi-Kershaw's inequality (2) and inequality (3).

Theorem 3. *Let $\theta(t)$ be defined by (10), $p(t) = t - \theta(t-1)$ in $(-\infty, \infty)$ and p^{-1} stand for the inverse function of p. Then inequalities*

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\big[(1-s)\psi\big(x + p^{-1}(s)\big)\big] \tag{15}$$

for $x \in (-s, \infty)$ *and*

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \geq \frac{\Gamma(\delta+1)}{\Gamma(\delta+s)} \exp\bigl[\psi\bigl(x+p^{-1}(s)\bigr)-\psi\bigl(\delta+p^{-1}(s)\bigr)\bigr] \tag{16}$$

for $x \in (\delta, \infty)$ *are valid for* $s \in (0,1)$*, where* $\delta > -s$ *and* $s \leq p^{-1}(s) \leq 1$*.*

Remark 2. By the software Mathematica 5.2, some values of $p(t)$ are calculated numerically:

$$\begin{aligned} &p(0.1) = -0.59286\cdots, && p(0.2) = -0.43195\cdots, && p(0.3) = -0.26780\cdots, \\ &p(0.4) = -0.10013\cdots, && p(0.5) = 0.071355\cdots, && p(0.6) = 0.24702\cdots, \\ &p(0.7) = 0.42726\cdots, && p(0.8) = 0.61249\cdots, && p(0.9) = 0.80322\cdots. \end{aligned}$$

This shows that $0.5 < p^{-1}(s) < 1$ for $s \in (0,1)$ approximately.

2. Proofs of theorems

Proof of Theorem 1. It is well known [1] that

$$\psi(x) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-xu}}{1-e^{-u}}\right) \mathrm{d}\, u \tag{17}$$

for $x > 0$.

For $a = b$, by (17), it is clear that

$$\ln F_{a,a,c}(x) = \psi(x+c) - \psi(x+a) = \int_0^\infty \frac{e^{-au} - e^{-cu}}{1-e^{-u}} e^{-xu} \,\mathrm{d}\, u$$

and

$$(-1)^i [\ln F_{a,a,c}(x)]^{(i)} = \int_0^\infty \frac{e^{-au} - e^{-cu}}{1-e^{-u}} u^i e^{-xu} \,\mathrm{d}\, u$$

for $i \in \mathbb{N}$. Therefore $F_{a,a,c}(x) \in \mathcal{L}[(-a,\infty)]$ if $a = b < c$ and $[F_{a,a,c}(x)]^{-1} \in \mathcal{L}[(-c,\infty)]$ if $c < a = b$.

For $b \neq a$, taking the logarithm of the function $F_{a,b,c}(x)$, differentiating and using (17) yields

$$\begin{aligned} \ln F_{a,b,c}(x) &= \psi(x+c) - \frac{\ln\Gamma(x+b) - \ln\Gamma(x+a)}{b-a} \\ &= \psi(x+c) - \frac{1}{b-a}\int_a^b \psi(x+t)\,\mathrm{d}\, t \\ &= \int_0^\infty \left(\frac{1}{b-a}\int_a^b e^{-tu}\,\mathrm{d}\, t - e^{-cu}\right)\frac{e^{-xu}}{1-e^{-u}}\,\mathrm{d}\, u \\ &= \int_0^\infty \left[\frac{e^{(c-a)u} - e^{(c-b)u}}{u(b-a)} - 1\right]\frac{e^{-(x+c)u}}{1-e^{-u}}\,\mathrm{d}\, u \end{aligned}$$

and

$$\begin{aligned} (-1)^i [\ln F_{a,b,c}(x)]^{(i)} &= \int_0^\infty \left[\frac{e^{(c-a)u} - e^{(c-b)u}}{u(b-a)} - 1\right]\frac{u^i e^{-(x+c)u}}{1-e^{-u}}\,\mathrm{d}\, u \\ &\triangleq \int_0^\infty [g_{c-a,c-b}(u) - 1]\frac{u^i e^{-(x+c)u}}{1-e^{-u}}\,\mathrm{d}\, u \end{aligned}$$

for $i \in \mathbb{N}$.

Let

$$h_{\alpha,\beta}(u) \triangleq g_{\alpha,\beta}(u) - 1 = \frac{[e^{\alpha u} - \alpha u] - [e^{\beta u} - \beta u]}{(\alpha - \beta)u} \tag{18}$$

for $u > 0$ and $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha \neq \beta$. It is easy to see that the function $e^t - t$ is decreasing in $(-\infty, 0)$ and increasing in $(0, \infty)$. See Figure 1. Consequently, if $\alpha \geq 0$ and $\beta \geq 0$, the function $h_{\alpha,\beta}(u)$ is positive in $(0, \infty)$; if $0 \geq \alpha$ and $0 \geq \beta$, the function $h_{\alpha,\beta}(u)$ is negative in $u \in (0, \infty)$. Let $\theta(t)$ be defined by (10). It is

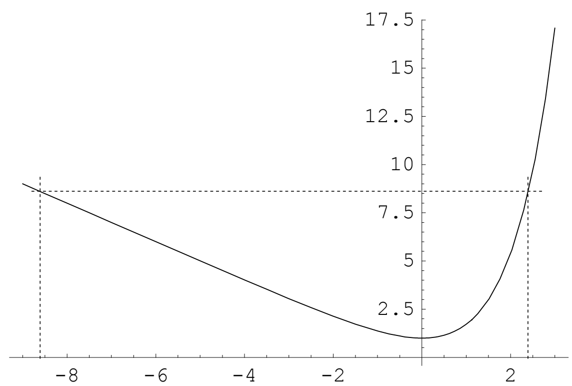

FIGURE 1. The graph of $e^t - t$ in $(-9, 3)$ pictured by MATHEMATICA 5.2

apparent that $t\theta(t) < 0$ for $\theta(t) \neq t$. As a result, if $\alpha \geq 0$ and $0 \geq \beta \geq \theta(\alpha)$, then $h_{\alpha,\beta}(u)$ is positive; if $\alpha \geq 0$ and $\beta \leq \theta(\alpha)$, then the function $h_{\alpha,\beta}(u)$ is negative in $(0, \infty)$; if $\alpha \leq 0$ and $0 \leq \beta \leq \theta(\alpha)$, then $h_{\alpha,\beta}(u)$ is negative; if $\alpha \leq 0$ and $\beta \geq \theta(\alpha)$, then the function $h_{\alpha,\beta}(u)$ is positive in $(0, \infty)$

From the positivity or negativity of the function $h_{\alpha,\beta}(u)$, the logarithmically completely monotonicity of the function $F_{a,b,c}(x)$ is obtained:

(1) If either $c - a \geq 0$ and $c - b \geq 0$ or $c - a \geq 0$ and $0 \geq c - b \geq \theta(c - a)$ or $c - a \leq 0$ and $c - b \geq \theta(c - a)$, then $F_{a,b,c}(x) \in \mathcal{L}[(-\rho, \infty)]$;

(2) If either $c - a \leq 0$ and $c - b \leq 0$ or $c - a \geq 0$ and $c - b \leq \theta(c - a)$ or $c - a \leq 0$ and $0 \leq c - b \leq \theta(c - a)$, then $[F_{a,b,c}(x)]^{-1} \in \mathcal{L}[(-\rho, \infty)]$.

The proof of Theorem 1 is complete. □

Proof of Theorem 2. For a and b being two constants, as $x \to \infty$, the following asymptotic formula is given in [1, p. 257 and p. 259]:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right) = 1 + O\left(\frac{1}{x}\right). \tag{19}$$

In [17], it was proved that $\psi(x) - \ln x + \alpha/x \in \mathcal{C}[(0, \infty)]$ if and only if $\alpha \geq 1$ and $\ln x - \alpha/x - \psi(x) \in \mathcal{C}[(0, \infty)]$ if and only if $\alpha \leq 1/2$. From this, it is deduced that

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \tag{20}$$

in $(0, \infty)$. Utilization of (19) and (20) leads to

$$\lim_{x \to \infty} F_{a,b,c}(x) = \lim_{x \to \infty} \left\{ \frac{\exp[\psi(x+c)]}{x} \left[1 + O\left(\frac{1}{x}\right)\right]^{1/(b-a)} \right\}$$

$$= \lim_{x\to\infty} \frac{\exp[\psi(x+c)]}{x}$$
$$= 1.$$

Hence, if $(a,b,c) \in D_1(a,b,c)$, then $F_{a,b,c}(x) > 1$ which is equivalent to inequality (13); if $(a,b,c) \in D_2(a,b,c)$, then $F_{a,b,c}(x) < 1$ which is equivalent to the reversed inequality of (13).

Let δ be a constant greater than $-\rho$. If $(a,b,c) \in D_1(a,b,c)$, then $F_{a,b,c}(x) \le F_{a,b,c}(\delta)$ in $x \in [\delta,\infty)$, which is equivalent to inequality (14); if $(a,b,c) \in D_2(a,b,c)$, then $F_{a,b,c}(x) \ge F_{a,b,c}(\delta)$ in $x \in [\delta,\infty)$, which is equivalent to the reversed inequality of (14). The proof of Theorem2 is complete. □

Proof of Corollary 3. Taking $a = 1$ and $b = s \in (0,1)$ in (13) and (14) leads to

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp[(1-s)\psi(x+c)] \tag{21}$$

for $x \in (-\rho,\infty)$ and

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \ge \frac{\Gamma(\delta+1)}{\Gamma(\delta+s)} \exp\{(1-s)[\psi(x+c)-\psi(\delta+c)]\} \tag{22}$$

for $x \in (\delta,\infty)$, where $\delta > -\rho$, $c \le 1$ and $c - s \ge \theta(c-1) \ge 0$. Since $\theta(t)$ is strictly decreasing in $t \in (-\infty,\infty)$, then the function $p(t)$ is strictly increasing in $t \in (-\infty,\infty)$. Thus, inequalities (21) and (22) validate for $c \ge p^{-1}(s)$ and $s \in (0,1)$. Since $\psi(x+c)$ is increasing and $\psi(x+c) - \psi(\delta+c)$ for $x > \delta$ is decreasing with respect to c, inequalities (21) and (22) have a best upper bound and a best lower bound $\exp\bigl[(1-s)\psi\bigl(x+p^{-1}(s)\bigr)\bigr]$ and $[\Gamma(\delta+b)/\Gamma(\delta+a)]^{1/(b-a)} \exp\bigl[\psi\bigl(x+p^{-1}(s)\bigr) - \psi\bigl(\delta+p^{-1}(s)\bigr)\bigr]$ respectively. The proof of Corollary 3 is complete. □

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.

[2] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[3] J. Bustoz and M. E. H. Ismail, *On gamma function inequalities*, Math. Comp. **47** (1986), 659–667.

[4] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 100. Available online at `http://jipam.vu.edu.au/article.php?sid=574`.

[5] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.

[6] T. Erber, *The gamma function inequalities of Gurland and Gautschi*, Scand. Actuar. J. **1960** (1961), 27–28.

[7] W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38** (1959/60), 77–81.

[8] J. D. Kečlić and P. M. Vasić, *Some inequalities for the gamma function*, Publ. Inst. Math. Beograd N. S. **11** (1971), 107–114.

[9] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.

[10] F. Qi, *A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality*, J. Comput. Appl. Math. (2006), in press. RGMIA Res. Rep. Coll. **9** (2006), no. 2, Art. 16. Available online at `http://rgmia.vu.edu.au/v9n2.html`.

[11] F. Qi, *A completely monotonic function involving divided differences of psi and polygamma functions and an application*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[12] F. Qi, *A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[13] F. Qi, *Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at `http://rgmia.vu.edu.au/v8n3.html`.

[14] F. Qi, *Monotonicity and logarithmic convexity for a class of elementary functions involving the exponential function*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art. 3. Available online at `http://rgmia.vu.edu.au/v9n3.html`.

[15] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 7, 1027–1034. Available online at `http://rgmia.vu.edu.au/v2n7.html`.

[16] F. Qi, *The best bounds in Kershaw's inequality and two completely monotonic functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[17] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 4. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[18] F. Qi, *Three-log-convexity for a class of elementary functions involving exponential function*, J. Math. Anal. Approx. Theory **1** (2006), no. 2, 100–103.

[19] F. Qi, J. Cao, and D.-W. Niu, *Four logarithmically completely monotonic functions involving gamma function and originating from problems of traffic flow*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art 9. Available online at `http://rgmia.vu.edu.au/v9n3.html`.

[20] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[21] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[22] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[23] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **80** (2006), 81–88.

[24] F. Qi, B.-N. Guo, and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436. RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 17. Available online at `http://rgmia.vu.edu.au/v8n2.html`.

[25] J. Wendel, *Note on the gamma function*, Amer. Math. Monthly **55** (1948), 563–564.

[26] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

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