# TANGENT SPHERE OF EDGE IN TETRAHEDRON

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ABSTRACT. In this short note, using one of Yang's Theorems, Lin-Zhu's conjecture for a geometric inequality with respect to the tangent sphere of the edge in tetrahedron is proved. In the final, an open problem on the $n-$dimension simplex is posed.

## 1. INTRODUCTION

For the tetrahedron, there is the inscribed sphere and circumscribed sphere with similarly the incircle and circumscricle of a triangle. In 1996, Z.-Ch. Lin and H.-F. Zhu [1] studied a new sphere in tetrahedron, that is the tangent sphere of the edge.

**Definition 1.1.** *If a sphere is tangent with every edge of a tetrahedron, then this sphere is called the tangent sphere of the edge of this tetrahedron.*

However, it is not certain that there is a tangent sphere of the edge in every tetrahedron. In 1985, Zh. Yang given a sufficient and necessary condition to determine whether a tetrahedron has one tangent sphere of the edge in [2] (see also [1]).

Let $\mathscr{P}$ denote tetrahedron $P_0P_1P_2P_3$ as follows.

**Theorem 1.1.** *$\mathscr{P}$ has the tangent sphere of the edge if and only if $x_i$ exist and satisfy $a_{ij} = x_i + x_j$ for $0 \le i < j \le 3$, where $P_iP_j = a_{ij}$.*

In [1], Lin and Zhu gave a formula to compute the radius and obtained several inequalities for the tangent sphere of the edge.

**Theorem 1.2.** *If $l$ is the radius of the tangent sphere of the edge of $\mathscr{P}$, then we have*

$$l^2 = \frac{(2x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3 \sum\limits_{0\le i<j\le 3} x_ix_j - (x_0^2x_1^2x_2^2 + x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2)}. \tag{1.1}$$

**Theorem 1.3.** *Let $V$ be the volume, $R$ the circumradius and $r$ the inradius of $\mathscr{P}$, respectively. Then for any $\mathscr{P}$ which has the tangent sphere of the edge, we have*

$$16l^2 \ge 2 \sum_{0\le i<j\le 3} x_ix_j - \sum_{i=0}^{3} x_i^2, \tag{1.2}$$

$$R^2 \ge 3l^2, \tag{1.3}$$

$$Vl \le \frac{1}{24}(a_{01}a_{02}a_{03}a_{12}a_{13}a_{23})^{\frac{2}{3}}, \tag{1.4}$$

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$$8r^2 l \le V \le \frac{8}{9}R^2 l \le \frac{8\sqrt{3}}{27}R^3, \tag{1.5}$$

*and*

$$16R^2 \ge 15R^2 + 3l^2 \ge a_{01}^2 + a_{02}^2 + a_{03}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2. \tag{1.6}$$

In the final of [1], the authors posed the following open problem (see also [3]).

**Conjecture 1.1.** *For any $\mathscr{P}$ which has the tangent sphere of the edge, prove or disprove that*

$$l^2 \ge 3r^2. \tag{1.7}$$

The main purpose of this paper is to give a proof of this conjecture.

## 2. The Proof of Conjecture 1.1

In order to prove Conjecture 1.1, we require several lemmas.

**Lemma 2.1.** *For triangle $ABC$, if $a = y + z$, $b = z + x$ and $c = x + y$, then*

$$r^2 = \frac{xyz}{x+y+z}. \tag{2.1}$$

*Proof.* Let $p$ be the half perimeter, $S$ the area, $a, b, c$ the side-lengths of triangle $ABC$. Then we find that

$$S = \sqrt{p(p-a)(p-b)(p-c)}, \tag{2.2}$$

and

$$r = \frac{S}{p}. \tag{2.3}$$

And for $x = p - a$, $y = p - b$, $z = p - c$. Lemma 2.1 is proved. □

**Lemma 2.2.** *If $x_i > 0$ for $0 \le i \le 3$, then*

$$\left(\frac{x_1+x_2+x_3}{x_1x_2x_3} + \frac{x_2+x_3+x_0}{x_2x_3x_0} + \frac{x_3+x_0+x_1}{x_3x_0x_1} + \frac{x_0+x_1+x_2}{x_0x_1x_2}\right) \cdot \frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3\sum\limits_{0\le i<j\le 3} x_ix_j - (x_0^2x_1^2x_2^2 + x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2)} \ge 6. \tag{2.4}$$

*Proof.* In fact

$$x_0^2x_1^2(x_2-x_3)^2 + x_0^2x_2^2(x_1-x_3)^2 + x_0^2x_3^2(x_1-x_2)^2 + x_1^2x_2^2(x_0-x_3)^2 + x_1^2x_3^2(x_0-x_2)^2 + x_2^2x_3^2(x_0-x_1)^2 \ge 0,$$

we have

$$x_0^2x_1^2x_2^2 + x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2 \ge \frac{2}{3}x_0x_1x_2x_3 \sum_{0\le i<j\le 3} x_ix_j.$$

Therefor, we get

$$2x_0x_1x_2x_3\sum_{0\le i<j\le 3} x_ix_j - (x_0^2x_1^2x_2^2 + x_1^2x_2^2x_3^2 + x_2^2x_3^2x_0^2 + x_3^2x_0^2x_1^2) \le \frac{4}{3}x_0x_1x_2x_3\sum_{0\le i<j\le 3} x_ix_j,$$

or

$$\begin{aligned}&\frac{4(x_0x_1x_2x_3)^2}{2x_0x_1x_2x_3\sum\limits_{0\le i<j\le 3}x_ix_j-(x_0^2x_1^2x_2^2+x_1^2x_2^2x_3^2+x_2^2x_3^2x_0^2+x_3^2x_0^2x_1^2)}\\ \ge&\frac{4(x_0x_1x_2x_3)^2}{\frac{4}{3}x_0x_1x_2x_3\sum\limits_{0\le i<j\le 3}x_ix_j}=\frac{3x_0x_1x_2x_3}{\sum\limits_{0\le i<j\le 3}x_ix_j}.\end{aligned} \tag{2.5}$$

It's easy to find the following identity

$$\frac{x_1+x_2+x_3}{x_1x_2x_3}+\frac{x_2+x_3+x_0}{x_2x_3x_0}+\frac{x_3+x_0+x_1}{x_3x_0x_1}+\frac{x_0+x_1+x_2}{x_0x_1x_2}=\frac{2\sum\limits_{0\le i<j\le 3}x_ix_j}{x_0x_1x_2x_3}. \tag{2.6}$$

From (2.5) and (2.6), we find immediately inequality (2.4). □

**Lemma 2.3.** *Let $V$ be the volume of $\mathscr{P}$, $r_0$, $r_1$, $r_2$ and $r_3$ the inradius, $S_0$, $S_1$, $S_2$ and $S_3$ the areas of triangle$P_1P_2P_3$, triangle$P_2P_3P_0$, triangle$P_3P_0P_1$ and triangle$P_0P_1P_2$, respectively. Then we have*

$$(S_0+S_1+S_2+S_3)^2\ge\frac{9V^2}{2}\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right), \tag{2.7}$$

*with the equality in* (2.7) *holds if and only if $\mathscr{P}$ is regular.*

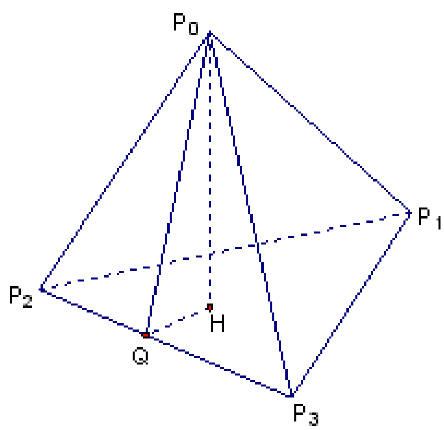


FIGURE 1.

*Proof.* As figure 1. Let $\alpha$, $\beta$ and $\gamma$ be the plane angles of the dihedral angels $P_0-P_2P_3-P_1$, $P_0-P_3P_1-P_2$ and $P_0-P_1P_2-P_3$, respectively. We draw $P_0H\perp P_1P_2P_3$ from $P_0$, $P_0\bigcap P_1P_2P_3=H$, and $P_0Q\perp P_2P_3$ from $P_0$, $P_0Q\bigcap P_2P_3=Q$, and link $QH$. Thus, the plane angles of the dihedral angels $P_0-P_2P_3-P_1$ is $\angle P_0QH$ and $P_0H=P_0Q\cdot\sin\alpha$. Hence,

$$P_0H\cdot P_2P_3=2S_1\sin\alpha=2\sqrt{(S_1+S_1\cos\alpha)(S_1-S_1\cos\alpha)}. \tag{2.8}$$

By the same argument as above, we also obtain

$$P_0H\cdot P_3P_1=2S_2\sin\beta=2\sqrt{(S_2+S_2\cos\beta)(S_2-S_2\cos\beta)}, \tag{2.9}$$

and

$$P_0H\cdot P_1P_2=2S_3\sin\gamma=2\sqrt{(S_3+S_3\cos\gamma)(S_3-S_3\cos\gamma)}. \tag{2.10}$$

From (2.8)–(2.10), together with $P_0H = \frac{3V}{S_0}$ and $\frac{S_0}{r_0} = \frac{1}{2}(P_1P_2 + P_2P_3 + P_3P_1)$, we get

$$\begin{aligned}\frac{3V}{r_0} =&\sqrt{(S_1 + S_1\cos\alpha)(S_1 - S_1\cos\alpha)}\\ &+\sqrt{(S_2 + S_2\cos\beta)(S_2 - S_2\cos\beta)}\\ &+\sqrt{(S_3 + S_3\cos\gamma)(S_3 - S_3\cos\gamma)}.\end{aligned} \tag{2.11}$$

By Cauchy's inequality and $S_0 = S_1\cos\alpha + S_2\cos\beta + S_3\cos\gamma$, we have

$$\begin{aligned}\left(\frac{3V}{r_0}\right)^2 \leq&(S_1 + S_1\cos\alpha + S_2 + S_2\cos\beta + S_3 + S_3\cos\gamma)\\ &\cdot(S_1 - S_1\cos\alpha + S_2 - S_2\cos\beta + S_3 - S_3\cos\gamma)\\ =&(S_1 + S_2 + S_3 + S_0)(S_1 + S_2 + S_3 - S_0)\\ =&(S_1 + S_2 + S_3)^2 - S_0^2,\end{aligned} \tag{2.12}$$

or

$$(S_1 + S_2 + S_3)^2 - S_0^2 \geq \left(\frac{3V}{r_0}\right)^2. \tag{2.13}$$

It's easy to know that equality in (2.13) holds if and only if $\frac{S_1+S_1\cos\alpha}{S_1-S_1\cos\alpha} = \frac{S_2+S_2\cos\beta}{S_2-S_2\cos\beta} = \frac{S_3+S_3\cos\gamma}{S_3-S_3\cos\gamma}$ or $\cos\alpha = \cos\beta = \cos\gamma$, also or $\alpha = \beta = \gamma$.

Similarly, yields

$$(S_2 + S_3 + S_0)^2 - S_1^2 \geq \left(\frac{3\,V}{r_1}\right)^2, \tag{2.14}$$

$$(S_3 + S_0 + S_1)^2 - S_2^2 \geq \left(\frac{3\,V}{r_2}\right)^2, \tag{2.15}$$

and

$$(S_0 + S_1 + S_2)^2 - S_3^2 \geq \left(\frac{3\,V}{r_3}\right)^2. \tag{2.16}$$

Summing (2.13) to (2.16), it follows that inequality (2.7) is true, and with the equality in (2.7) holds if and only if $\mathscr{P}$ is regular.

Thus, the proof of Lemma 2.3 is completed. □

**Remark 2.1.** *Inequality* (2.7) *is obtained by X.-Zh. Yang in* [4].

**Theorem 2.1.** *In $\mathscr{P}$ which has the tangent sphere of the edge, we have that inequality* (1.7) *holds.*

*Proof.* For $\mathscr{P}$, from Lemma 2.1, Theorem 1.2 and Lemma 2.2, we have

$$\left(\frac{1}{r_0^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2}\right) l^2 \geq 6. \tag{2.17}$$

By Lemma 2.3, and $r = \frac{3\,V}{S_0+S_1+S_2+S_3}$, it follows

$$l^2 \geq \frac{6}{\frac{1}{r_0^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2}} \geq \frac{27\,V^2}{(S_0 + S_1 + S_2 + S_3)^2} \geq 3r^2. \tag{2.18}$$

Therefor, Theorem 2.1 is proved. □

**Remark 2.2.** *Combining* (1.3) *and* (1.7), *we have*

$$R^2 \geq 3l^2 \geq 9r^2. \tag{2.19}$$

## 3. An Open Problem

As a generalization of the tetrahedron, we should consider the tangent sphere of the edge in $n-$dimensional simplex in this section.

The following definition and theorems of the $n-$dimensional simplex are similar with the tetrahedron in [5].

**Definition 3.1.** *If a sphere is tangent with every edge of a $n-$dimensional simplex, then this sphere is the tangent sphere of the edge of this $n-$dimensional simplex.*

**Theorem 3.1.** *Suppose that $P_iP_j = a_{ij}$ is the edges of $n-$dimensional simplex $P_0P_1P_2 \cdots P_n$, then it has a tangent sphere of the edge if and only if $x_i$ exist and satisfy $a_{ij} = x_i + x_j$ for $0 \le i \ne j \le n$.*

**Theorem 3.2.** *Let $l$ be the radius of the tangent sphere of the edge of $n-$dimensional simplex $P_0P_1P_2 \cdots P_n$. Then*

$$l^2 = -\frac{D_1}{2D_2}, \tag{3.1}$$

*where*

$$D_1 = \begin{vmatrix} -2x_0^2 & 2x_0x_1 & \cdots & 2x_0x_{n-1} \\ 2x_0x_1 & -2x_1^2 & \cdots & 2x_1x_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 2x_0x_{n-1} & 2x_1x_{n-1} & \cdots & -2x_{n-1}^2 \end{vmatrix},$$

*and*

$$D_2 = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \cdot & \cdots & \cdot \\ \vdots & \vdots & D_1 & \vdots \\ 1 & \cdot & \cdots & \cdot \end{vmatrix}.$$

Now, we pose an open problem on the inscribed sphere and the tangent sphere of the edge $n-$dimensional simplex.

**Conjecture 3.1.** *In $n-$dimensional simplex $P_0P_1P_2 \cdots P_n$ which has the tangent sphere of the edge, prove or disprove that*

$$l^2 \ge nr^2. \tag{3.2}$$

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