RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 3, (2008), Solution No. 1

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Received: 2 November, 2008

Problem. *Let $a, b > 0$. Prove that*

$$\int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)\,\mathrm{d}t \geq \mathrm{B}(a,b)\Gamma\Big(\frac{a}{a+b}\Big)\,, \tag{1}$$

where B, Γ *denote the Euler's functions of the first and second kind respectively.*

Proof. Denote $\mathscr{L}, \mathscr{R}$ as the left, respectively the right sides of the posed inequality (1). We have

$$\begin{aligned} \mathscr{L} &= \int_0^1 t^{a-1}(1-t)^{b-1}\left\{\int_0^\infty \mathrm{e}^{-x}x^{t-1}\mathrm{d}x\right\}\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-x}x^{-1}\left\{\int_0^1 t^{a-1}(1-t)^{b-1}x^t\,\mathrm{d}t\right\}\mathrm{d}x \\ &= \mathrm{B}(a,b)\int_0^\infty \mathrm{e}^{-x}x^{-1}\,{}_1F_1\big(a;a+b;\ln x\big)\,\mathrm{d}x \end{aligned} \tag{2}$$

by [1, **3.383.** (1)]. Here ${}_1F_1$ stands for the confluent hypergeometric function. Consider now a Luke–type inequality [2, Theorem 16, Eqs. 5.5–7] (see also [3, Eq. (16)]):

$$ {}_1F_1\big(\alpha;\rho;x\big) \geq \mathrm{e}^{\frac{\alpha}{\rho}\,|x|} \qquad \big(\rho \geq \alpha > 0, x \in \mathbb{R}\big). \tag{3} $$

Applying (3) to the hypergeometric expression in the integrand of (2), we get

$$ \begin{aligned} \mathscr{L} &\geq \mathrm{B}(a,b) \int_0^\infty \mathrm{e}^{-x} x^{-1}\, \mathrm{e}^{\frac{a}{a+b}\,|\ln x|}\,\mathrm{d}x \\ &\geq \mathrm{B}(a,b) \int_0^\infty \mathrm{e}^{-x} x^{-1}\, \mathrm{e}^{\frac{a}{a+b}\,\ln x}\,\mathrm{d}x \\ &= \mathrm{B}(a,b) \int_0^\infty \mathrm{e}^{-x} x^{\frac{a}{a+b}-1}\,\mathrm{d}x \equiv \mathscr{R}\,. \end{aligned} $$

The proof is complete. □

Lastly, we can remark that the lower bound (1) mainly improves the obvious estimate $\mathscr{L} \geq \mathrm{B}(a,b)$.

References

[1] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Intgerals, Series and Products* (Corrected and Enlarged Edition prepared by A.Jeffery and D. Zwillinger), Sixth ed. Academic Press, New York, 2000.

[2] Y.L. Luke, Inequalities for generalized hypergeometric functions, *J. Approx. Theory* **5**(1972) 41–65.

[3] T.K. Pogány and H.M. Srivastava, Some Mathieu–type series associated with the Fox–Wright function, Computers and Mathematics with Applications (2008), `doi:10.1016/j.camwa.2008.07.016`.