RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 4, (2008), Solution No. 1

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Problem 4. Let a, b, c be the sides of $\triangle ABC$. Prove that

$$\sum \frac{a}{b+c}\left(\tan^4 \frac{B}{2} + \tan^4 \frac{C}{2}\right) \geq \frac{1}{3}$$

Solution. First, let us prove a lemma.

Lemma 1. *For any $a, b, c, x, y, z > 0$ we have*

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \geq \sqrt{3(xy+yz+zx)}$$

Proof. . Because the inequality is homogeneous in x, y, z we can assume that $x + y + z = 1$. We rewrite the inequality as follows

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a}{b+c}x + \frac{b}{c+a}y + \frac{c}{a+b}z + \sqrt{3(xy+yz+zx)}$$

But then we can apply the **Cauchy-Schwarz Inequality** so that to obtain

$$\frac{a}{b+c}x+\frac{b}{c+a}y+\frac{c}{a+b}z+\sqrt{3(xy+yz+zx)}$$
$$\leq\sqrt{\sum\left(\frac{a}{b+c}\right)^2}\sqrt{\sum x^2}+\sqrt{\frac{3}{4}}\sqrt{\sum xy}+\sqrt{\frac{3}{4}}\sqrt{\sum xy}$$
$$\leq\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}}\sqrt{\sum x^2+2\sum xy}=\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}}$$

Thus, we are left with the inequality

$$\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}}\leq\sum\frac{a}{b+c}$$

But this one is equivalent to

$$\sum\frac{ab}{(c+a)(c+b)}\geq\frac{3}{4}$$
$$\Leftrightarrow 4\sum ab(a+b)\geq 3(a+b)(b+c)(c+a)$$
$$\Leftrightarrow a^2b+b^2c+c^2a=ab^2+bc^2+ca^2\geq 6abc$$

which is true from the **AM-GM Inequality**. □

Now let us turn back to the solution of the problem, let

$$\tan^4\frac{A}{2}=x,\tan^4\frac{B}{2}=y,\tan^4\frac{C}{2}=z$$

By the **Holder's Inequality**, we have

$$27(xy+yz+zx)=27\sum\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^4\geq\left(\sum\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)\right)^4=1$$

According to the **lemma** we have

$$\sum\frac{a}{b+c}\left(\tan^4\frac{B}{2}+\tan^4\frac{C}{2}\right)\geq\sqrt{3(xy+yz+zx)}=\frac{2}{3}$$

Equality holds if and only if $A=B=C=60^o$.

References

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[3] Andreescu T., Cartoaje V., Dospinescu G., Lascu M., *Old and new inequalities*, GIL Publ. House, Romania, 2004.