RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 6, (2008), Solution No. 1

S.M. Sitnik
Voronezh Militia Institute
Voronezh, Russia
Email: `box2008in@gmail.com`

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1. First consider this interesting inequality due to Prof. Simic without weights:

$$\prod_{k=1}^{n}(1+a_k) \le \left(1+\frac{(a_1+a_n)^2}{4a_1a_n}G(a_1,\dots,a_n)\right)^n \tag{1}$$

We shall prove more that

$$\prod_{k=1}^{n}(1+a_k) \le \left(1+\frac{(a_1+a_n)^2}{4a_1a_n}H(a_1,\dots,a_n)\right)^n, \tag{2}$$

$H(a_1,\dots,a_n)$—harmonic mean. (2) is better than (1) because $H \le G$.

To prove (2) first apply the AGM inequality

$$\left(\prod_{k=1}^{n}(1+a_k)\right)^{\frac{1}{n}} \le \frac{\sum_{k=1}^{n}(1+a_k)}{n} = 1+\frac{1}{n}\sum_{k=1}^{n}a_k = 1+A(a_1,\dots,a_n).$$

So we have to prove

$$A(a_1,\dots,a_n) \le \frac{(a_1+a_n)^2}{4a_1a_n}H(a_1,\dots,a_n), A(a_1,\dots,a_n)\cdot\frac{1}{H(a_1,\dots,a_n)} \le \frac{(a_1+a_n)^2}{4a_1a_n}.$$

But this is a special case of P. Schweitzer inequality [1]

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{a_k}\right) \le \frac{(a_{min}+a_{max})^2}{4a_{min}a_{max}}.$$

That's all.

So we proved a bit more than original inequality:
for any positive numbers it is valid that

(3) $$\prod_{k=1}^{n}(1+a_k) \le \left(1+\frac{(a_{min}+a_{max})^2}{4a_{min}a_{max}}H(a_1,\ldots,a_n)\right)^n.$$

2. Now consider this inequality with weights

(4) $$\prod_{k=1}^{n}(1+a_k)^{p_k} \le \left(1+\frac{(a_1+a_n)^2}{4a_1a_n}\right)\prod_{k=1}^{n} {a_k}^{p_k}, p_k \ge 0, \sum_{k=1}^{n} p_k = 1.$$

Repeat the previous proof and apply the weighted AGM inequality

$$\prod_{k=1}^{n}(1+a_k)^{p_k} \le \sum_{k=1}^{n} p_k(1+a_k) = \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} p_k a_k = 1 + \sum_{k=1}^{n} p_k a_k.$$

So we have to prove

$$\sum_{k=1}^{n} p_k a_k \le \frac{(a_1+a_n)^2}{4a_1a_n}\prod_{k=1} {a_k}^{p_k},$$

Again we prove more changing geometric mean by smaller harmonic mean

$$\sum_{k=1}^{n} p_k a_k \le \frac{(a_1+a_n)^2}{4a_1a_n}\left(\sum_{k=1}\frac{p_k}{a_k}\right)^{-1}.$$

So we need to prove

(5) $$\sum_{k=1}^{n} p_k a_k \left(\sum_{k=1}^{n}\frac{p_k}{a_k}\right) \le \frac{(a_1+a_n)^2}{4a_1a_n}$$

It is true that $a_1 = a_{min}, a_n = a_{max}$. But then this is exactly the famous Kantorovich inequality [2]-[3] and this completes the proof.

So again we proved a bit more than original inequality:

for any positive numbers it is valid that

$$\prod_{k=1}^{n}(1+a_k)^{p_k} \le \left(1+\frac{(a_{min}+a_{max})^2}{4a_{min}a_{max}}\right)\left(\sum_{k=1}^{n}\frac{p_k}{a_k}\right)^{-1}. \tag{6}$$

References

[1] D.S.Mitrinović (In cooperation with P.M.Vasić) *Analytic Inequalities*, Springer, 1970.

[2] R.A. Horn, Ch. R. Johnson *Matrix Analysis*, Cambridge University Press, 1986.

[3] Alpargu G. The Kantorovich inequality, with some extensions and with some statistical applications. Thesis, Department of Mathematics and Statistics McGill University, Montreal,Canada, 1996.