RESEARCH GROUP IN MATHEMATICAL INEQUALITIES AND APPLICATIONS

PROBLEM CORNER

Problem 10, (2009), Solution No. 1

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Firstly, for convenience, we denote the angles again.

Let $\widehat{A_1} = \angle 5, \widehat{A_2} = \angle 4, \widehat{B_1} = \angle 1, \widehat{B_2} = \angle 6, \widehat{C_1} = \angle 3, \widehat{C_2} = \angle 2.$

The problems of Jian Liu can be rewritten in the form:

$$\cos B_1 \cos C_2 + \cos C_1 \cos A_2 + \cos A_1 \cos B_2 \leq \frac{9}{4}$$

$$\cos B_2 \cos C_1 + \cos C_2 \cos A_1 + \cos B_1 \cos A_2 \leq \frac{9}{4}$$

We are going to prove the generalization results of Jian Liu:

Theorem: For any non-obtuse triangle ABC, arbitrary point P in triangle, and three numbers x, y, z the following inequalities hold:

$$x \cos B_1 \cos C_2 + y \cos C_1 \cos A_2 + z \cos A_1 \cos B_2 \leq \frac{(xy + yz + zx)^2}{4xyz}$$

$$x \cos B_2 \cos C_1 + y \cos C_2 \cos A_1 + z \cos B_1 \cos A_2 \leq \frac{(xy + yz + zx)^2}{4xyz}$$

Proof: Firstly,we express and prove the following lemma:

Lemma: In any triangle $ABC$and three numbers x, y, z,we have:

$$x\cos A + y\cos B + z\cos C \leq \frac{x^2y^2 + y^2z^2 + z^2x^2}{2xyz}$$

Proof: We choose three unit vectors $\vec{e_a}, \vec{e_b}, \vec{e_c}$ on BC, CA, AB, such that:

$$\vec{e_a} \upuparrows \overrightarrow{BC}, e_b \upuparrows \overrightarrow{CA}, e_c \upuparrows \overrightarrow{AB}$$

For any u, v, w we have:

$$(u\vec{e_a} + \vec{e_b} + w\vec{e_c})^2 \geq 0$$
$$\Leftrightarrow u^2 + v^2 + w^2 + 2uv\vec{e_a}\vec{e_b} + 2vw\vec{e_b}\vec{e_c} + 2wu\vec{e_c}\vec{e_a} \geq 0$$
$$\Leftrightarrow u^2 + v^2 + w^2 - 2uv\cos C - 2vw\cos A - 2wu\cos B \geq 0$$
$$\Leftrightarrow \frac{u^2 + v^2 + w^2}{2} \geq vw\cos A + wu\cos B + uv\cos C$$

Letting $x = vw, y = wu, z = uv$ we get:

$$x\cos A + y\cos B + z\cos C \leq \frac{x^2y^2 + y^2z^2 + z^2x^2}{2xyz}$$

The proof for the lemma is completed.
Now back to our problem.
It is easy to check that:

$$2x\cos B_1\cos C_2 = x(\cos(B_1 - C_2) + \cos(B_1 + C_2)) \leq x + x\cos(B_1 + C_2).$$

Similarly, we get

$$2(x\cos B_1\cos C_2 + y\cos C_1\cos A_2 + z\cos A_1\cos B_2)$$
$$\leq x + y + z + [x\cos(B_1 + C_2) + y\cos(C_1 + A_2) + z\cos(A_1 + B_2)].$$

Note that: $(B_1 + C_2) + (C_1 + A_2) + (A_1 + B_2) = \pi$.

Then if we let $(B_1 + C_2) = X, (C_1 + A_2) = Y, A_1 + B_2 = Z$ we get another triangle XYZ. Applying our lemma for the triangle XYZ and three positive numbers x, y, z we have:

$$x\cos X + y\cos Y + z\cos Z \leq \frac{x^2y^2 + y^2z^2 + z^2x^2}{2xyz}$$

or equivalently:

$$x\cos(B_1 + C_2) + y\cos(C_1 + A_2) + z\cos(A_1 + B_2) \leq \frac{x^2y^2 + y^2z^2 + z^2x^2}{2xyz}$$

Then:

$$2(x\cos B_1\cos C_2 + y\cos C_1\cos A_2 + z\cos A_1\cos B_2) \le x + y + z + \frac{x^2y^2 + y^2z^2 + z^2x^2}{2xyz}$$

Therefore:

$$x\cos B_1\cos C_2 + y\cos C_1\cos A_2 + z\cos A_1\cos B_2 \le \frac{(xy + yz + zx)^2}{4xyz}$$

Using the same method we can easily prove the 2nd problem:

$$x\cos B_2\cos C_1 + y\cos C_2\cos A_1 + z\cos B_1\cos A_2 \le \frac{(xy + yz + zx)^2}{4xyz}$$

Equality occurs iff the triangle is equilateral and $x = y = z$. The proof for the theorem is completed.

Remark. The following inequality is another application of our lemma:

$$x\cos^2\frac{A}{2} + y\cos^2\frac{B}{2} + z\cos^2\frac{C}{2} \le \frac{(xy + yz + zx)^2}{4xyz}$$

Moreover, using our method we can prove the inequality for two triangles as follows:

For two triangles $\triangle A_1B_1C_1, \triangle A_2B_2C_2$ and three numbers x, y, z we have:

$$x\cos\frac{A_1}{2}\cos\frac{A_2}{2} + y\cos\frac{B_1}{2}\cos\frac{B_2}{2} + z\cos\frac{C_1}{2}\cos\frac{C_2}{2} \le \frac{(xy + yz + zx)^2}{4xyz}$$